

# **REPRESENTATIONS OF RELATIVELY FREE PROFINITE SEMIGROUPS, IRREDUCIBILITY, AND ORDER PRIMITIVITY**

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**ABSTRACT.** We establish that, under certain closure assumptions on a pseudovariety of semigroups, the corresponding relatively free profinite semigroups freely generated by a non-singleton finite set act faithfully on their minimum ideals. As applications, we enlarge the scope of several previous join irreducibility results for pseudovarieties of semigroups, which turn out to be even join irreducible in the lattice of pseudovarieties of ordered semigroups, so that, in particular, they are not generated by proper subpseudovarieties of ordered semigroups. We also prove the stronger form of join irreducibility for the Krohn-Rhodes complexity pseudovarieties, thereby solving a problem proposed by Rhodes and Steinberg.

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## 1. INTRODUCTION

Finite semigroups appear naturally in computer science as transition semigroups of finite automata, which makes them into algebraic recognition devices for regular languages. A more direct connection is obtained by associating with a regular language its syntactic semigroup, namely the quotient of the free semigroup on the underlying alphabet in which two words are identified if they appear in the same contexts with respect to the language. The syntactic semigroup is thus naturally ordered by comparison of the contexts of words. The properties of classes of regular languages that are captured by algebraic properties of their syntactic semigroups, respectively of their ordered syntactic semigroups, have been characterized in terms of closure properties with respect to certain natural combinatorial operators. Such classes of regular languages are known as varieties, respectively positive varieties, of regular languages. The algebraic counterparts are the so-called pseudovarieties of semigroups [18], respectively pseudovarieties of ordered semigroups [32], both characterized by natural algebraic closure properties.

Thus, since the syntactic semigroup can be effectively computed, to determine whether a given regular language belongs to a given variety becomes a decision problem on the corresponding pseudovariety, namely to determine whether a given finite semigroup belongs to it. Natural operators on varieties of languages correspond to natural operators on pseudovarieties of semigroups. But, such operators are often defined in terms of generators, rather than by characteristic properties of their members. The expression of a pseudovariety in terms of simpler pseudovarieties involving those operators, besides having structural significance, sometimes leads to decision procedures for the membership problem. However, whether the existence of such procedures may be inferred depends on the operators involved. In fact, the membership problem for pseudovarieties admitting decompositions in terms of several operators may be rather difficult [21], and even undecidable [1, 12].

One particularly simple operator on pseudovarieties is the join, in the lattice of pseudovarieties. The existence of nontrivial join decompositions, in the strict sense, or more generally of nontrivial join covers, has been investigated by several authors. Some pseudovarieties admit non-obvious join decompositions [3, Chapter 9], whereas some have been shown not to admit any nontrivial join covers [26, 40, 41]. Basically, two approaches have been devised to handle this problem: the syntactical approach, through pseudoidentities, which may be used to define pseudovarieties [37, 28, 33]; and the structural approach, through the investigation of special structural properties of generators of the pseudovarieties, such as the Kovács-Newman property [41, Section 7.4]. Recently, we have improved the results of [26] using a

variant of the syntactical approach used in [8]. In the present paper, we combine the two approaches to obtain results that cover and improve most of the previous join irreducibility results found in the literature. We are also able to prove join irreducibility of the Krohn-Rhodes complexity pseudovarieties, which solves part of [41, Problem 43]. Furthermore, our approach yields yet a finer result: the pseudovarieties in question are in fact join irreducible in the larger lattice of pseudovarieties of ordered semigroups.

Pseudoidentities are formal equalities between members of relatively free profinite semigroups. Relatively free profinite semigroups have a rich and often mysterious structure. Like any semigroup with a minimum ideal, they act by left and right multiplication on their minimum ideals. A key property considered in [41] in the finite case, not just in connection with the join irreducibility question, is that both such representations be faithful. A somewhat weaker property, which has apparently not been considered before, and is much easier to establish, is that the action of elements outside the minimum ideal  $K$  can be distinguished, among themselves and in comparison with those of  $K$ , by their action by multiplication on each side of  $K$ . Combined with an additional closure property involving a certain Rees matrix extension, we show that this is enough to prove join irreducibility of the pseudovariety in the lattice of pseudovarieties of ordered semigroups. Alternatively, the assumption that the corresponding variety of languages is closed under concatenation also leads to the same conclusion.

A key technique in this paper is thus to consider the left and right actions of a profinite semigroup on its minimum ideal, that is, the natural representation of the semigroup in the translational hull of the minimum ideal. This combines the discrete and topological cases considered, respectively in [41, Section 5.5.1] and [15, Chapter 4]. If both left and right components of that natural representation are faithful, then the minimum ideal is reductive and it follows that its translational hull is a profinite semigroup.

We further establish the faithfulness of both representations for relatively free profinite semigroups on several pseudovarieties. On the other hand, we show that a profinite semigroup for which both representations are faithful admits no nontrivial closed partial order compatible with multiplication. An application is that, if, for the finitely generated free profinite semigroup over a pseudovariety on an arbitrarily large number of generators, both representations are faithful, then the pseudovariety is not generated by any proper subpseudovariety of ordered semigroups. To establish such a property was in fact the original motivation for the present work. Although this property is a consequence of join irreducibility in the lattice of pseudovarieties of ordered semigroups, the result opens up the potential range of applications, as it requires no closure properties on the pseudovariety, unlike our results on join irreducibility.

A summary of the main applications of our techniques and related problems which are left open is given in a table at the end of Section 9.

## 2. PRELIMINARIES

This paper owes much to the book [41], which facilitated the access and further improved many key ideas in finite semigroup theory which were previously dispersed through many research papers. Another basic reference in the area is [3]. The reader is referred to those books for undefined notions and notation, as well as general background in the area.

Throughout this paper, (locally) compact spaces are assumed to be Hausdorff.

**2.1. Some pseudovarieties and operations on them.** For the reader's convenience, the following is a list of pseudovarieties of semigroups that play a role in this paper. Each item in the list is described by a characteristic property of its elements as well as by a basis or bases of pseudoidentities.

- S: all,  $\llbracket x = x \rrbracket$ .
- I: trivial,  $\llbracket x = y \rrbracket$ .
- Sl: semilattices,  $\llbracket x^2 = x, xy = yx \rrbracket$ .
- N: nilpotent,  $\llbracket x^\omega = 0 \rrbracket$ .
- D: definite,  $\llbracket xy^\omega = y^\omega \rrbracket$ .
- $D_n$ : definite of degree  $n$ ,  $\llbracket xy_1 \cdots y_n = y_1 \cdots y_n \rrbracket$ .
- K: reverse definite,  $\llbracket x^\omega y = x^\omega \rrbracket$ .
- $K_n$ : reverse definite of degree  $n$ ,  $\llbracket x_1 \cdots x_n y = x_1 \cdots x_n \rrbracket$ .
- Ll: locally trivial,  $\llbracket x^\omega y x^\omega = x^\omega \rrbracket$ .
- LSl: local semilattices,  $\llbracket x^\omega y x^\omega y x^\omega = x^\omega y x^\omega, x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega \rrbracket$ .
- LZ: left zero,  $\llbracket xy = x \rrbracket = K_1$ .
- RZ: right zero,  $\llbracket xy = y \rrbracket = D_1$ .
- RB: rectangular bands,  $\llbracket x^2 = x, xyx = x \rrbracket$ .
- B: bands,  $\llbracket x^2 = x \rrbracket$ .
- A: aperiodic,  $\llbracket x^{\omega+1} = x^\omega \rrbracket$ .
- G: groups,  $\llbracket x^\omega = 1 \rrbracket$ .
- $Ab_n$ : Abelian groups of exponent dividing  $n$ ,  $\llbracket x^n = 1, xy = yx \rrbracket$ .
- ReG: rectangular groups,  $\llbracket xy^\omega x^\omega = x \rrbracket$ .
- CS: completely simple,  $\llbracket x(yx)^\omega = x \rrbracket$ .
- CR: completely regular,  $\llbracket x^{\omega+1} = x \rrbracket$ .
- J:  $\mathcal{J}$ -trivial,  $\llbracket (xy)^\omega = (xy)^\omega x = (yx)^\omega \rrbracket$ .
- DA: regular  $\mathcal{D}$ -classes are aperiodic subsemigroups,  
 $\llbracket ((xy)^\omega x)^2 = (xy)^\omega x \rrbracket = \llbracket (xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega, x^{\omega+1} = x^\omega \rrbracket$ .
- DO: regular  $\mathcal{D}$ -classes are rectangular groups,  
 $\llbracket (xy)^\omega (yx)^\omega (yx)^\omega = (xy)^\omega \rrbracket$ .
- DS: regular  $\mathcal{D}$ -classes are subsemigroups,  
 $\llbracket ((xy)^\omega x)^{\omega+1} = (xy)^\omega x \rrbracket = \llbracket ((xy)^\omega (yx)^\omega (xy)^\omega)^\omega = (xy)^\omega \rrbracket$ .

It is well known that  $\mathbf{D} = \bigcup_{n \geq 1} \mathbf{D}_n$  and  $\mathbf{K} = \bigcup_{n \geq 1} \mathbf{K}_n$ . For a pseudovariety  $\mathbf{H}$  of groups,  $\bar{\mathbf{H}}$  denotes the pseudovariety of all finite semigroups all of whose subgroups belong to  $\mathbf{H}$ .

A pseudovariety of semigroups  $\mathbf{V}$  is said to be *monoidal* if it is generated by its monoids; equivalently, whenever a semigroup  $S$  belongs to  $\mathbf{V}$ , so does the smallest monoid  $S^1$  containing  $S$ . This is the case, for example for the pseudovarieties  $\mathbf{DS}$ ,  $\mathbf{DO}$ ,  $\mathbf{CR}$ , and for those of the form  $\bar{\mathbf{H}}$ . Moreover, the intersection of monoidal pseudovarieties is again a monoidal pseudovariety.

Recall that a relational morphism of semigroups is a relation  $\mu : S \rightarrow T$  whose domain is  $S$  and such that  $\mu$  is a subsemigroup of  $S \times T$ . In particular, a homomorphism of semigroups is a relational morphism. For a given pseudovariety of semigroups  $\mathbf{U}$ , a relational morphism  $\mu : S \rightarrow T$  is a *U-relational morphism* if, for every idempotent  $e \in T$ , the subsemigroup  $\mu^{-1}(e) = \{s \in S : (s, e) \in \mu\}$  of  $S$  belongs to  $\mathbf{U}$ . A *U-homomorphism* is a homomorphism which is also a U-relational morphism.

The Mal'cev product  $\mathbf{U} \mathbin{\overline{\cap}} \mathbf{V}$  of the pseudovarieties  $\mathbf{U}$  and  $\mathbf{V}$  may be defined as the pseudovariety generated by the finite semigroups  $S$  for which there is a U-homomorphism  $S \rightarrow T$  into some  $T \in \mathbf{V}$ . Equivalently,  $\mathbf{U} \mathbin{\overline{\cap}} \mathbf{V}$  consists of all finite semigroups  $S$  for which there is some U-relational morphism  $S \rightarrow T$  into some  $T \in \mathbf{V}$ .

For a semigroup  $S$ , let  $S^I$  be the monoid that is obtained from  $S$  by adding a new neutral element, even  $S$  already has one. Note that, if  $S = S^1$ , then  $S^I$  is isomorphic to the subsemigroup  $S \times \{0\} \cup \{(1, 1)\}$  of  $S \times U_1$ , where  $U_1 = \{0, 1\}$  is a semilattice under the usual product.

Let  $\mathbf{U}$  and  $\mathbf{V}$  be monoidal pseudovarieties and suppose that  $\mathbf{U}$  contains **Sl**. Suppose that  $\mu : S \rightarrow T$  is a U-relational morphism into a semigroup  $T \in \mathbf{V}$ . Let  $\nu : S^I \rightarrow T^1$  be the relation given by  $\nu = \mu \cup \{(I, 1)\}$ . Then,  $\nu$  is a U-relational morphism into a semigroup from  $\mathbf{V}$ . Hence,  $S^I$  belongs to  $\mathbf{U} \mathbin{\overline{\cap}} \mathbf{V}$ , which shows that this pseudovariety is also monoidal.

It is well known that the Mal'cev product satisfies the following law [41, Exercise 2.3.20]:

$$\mathbf{U} \mathbin{\overline{\cap}} (\mathbf{V} \mathbin{\overline{\cap}} \mathbf{W}) \subseteq (\mathbf{U} \mathbin{\overline{\cap}} \mathbf{V}) \mathbin{\overline{\cap}} \mathbf{W}.$$

In particular, if  $\mathbf{U}$  is a fixed point of the operator  $\mathbf{U} \mathbin{\overline{\cap}} \_$ , then this operator is idempotent and so its fixed points are precisely the pseudovarieties of the form  $\mathbf{U} \mathbin{\overline{\cap}} \mathbf{V}$ , where  $\mathbf{V}$  is an arbitrary pseudovariety. Moreover, since  $\mathbf{U} \mathbin{\overline{\cap}} \bigcap_{i \in I} \mathbf{V}_i \subseteq \bigcap_{i \in I} (\mathbf{U} \mathbin{\overline{\cap}} \mathbf{V}_i)$ , the set of fixed points of the operator  $\mathbf{U} \mathbin{\overline{\cap}} \_$  is then a complete meet subsemilattice of the lattice of all pseudovarieties of semigroups.

Examples of pseudovarieties  $\mathbf{U}$  satisfying the equation  $\mathbf{U} \mathbin{\overline{\cap}} \mathbf{U} = \mathbf{U}$  of particular interest in this paper are **A**, **DA**, **B**, **D**, **K**, **LI**, **LZ**, **RB**, and **RZ**, although several others in the above list have the same property.

We adopt the following definition of semidirect product in the semigroup setting. Given semigroups  $S$  and  $T$ , and a monoid homomorphism from  $T^1$  into the monoid of endomorphisms of  $S$ , the associated *semidirect product*  $S * T$  consists of the set  $S \times T$  with the operation  $(s_1, t_1)(s_2, t_2) = (s_1 {}^{t_1}s_2, t_1 t_2)$ , where  ${}^{t_1}s_2$  denotes the image of  $s_2$  under the endomorphism of  $S$  corresponding to  $t_1$ . The *semidirect product* of the pseudovarieties of semigroups  $\mathbf{V}$  and  $\mathbf{W}$  is the pseudovariety generated by all semigroups of the form  $S * T$  with  $S \in \mathbf{V}$  and  $T \in \mathbf{W}$ . This produces an associative operation on pseudovarieties of semigroups but the reader is warned that it is not the definition adopted by some authors. See [41, Example 2.4.24] for a comparison with the definition adopted in that book. With our definition, the semidirect product of monoidal pseudovarieties is monoidal [3, Exercise 10.2.4].

**2.2. The de Bruijn encoding.** For a pseudovariety of semigroups  $\mathbf{V}$  and a finite set  $A$ ,  $\overline{\Omega}_A \mathbf{V}$  denotes the pro- $\mathbf{V}$  semigroup freely generated by  $A$ . Elements of  $\overline{\Omega}_A \mathbf{V}$  will in general be called *pseudowords*.

Let  $V$  be a pseudovariety containing  $D_n$ . For a pseudoword  $w \in \overline{\Omega}_A V$ , denote by  $t_n(w)$  the longest suffix of  $w$  of length  $|w|$  at most  $n$ . By looking at the natural projection  $\overline{\Omega}_A V \rightarrow \overline{\Omega}_A D_n$ , one sees immediately that there is only one such suffix, which justifies the notation. Dually, under the hypothesis that  $V$  contains  $K_n$ ,  $i_n(w)$  denotes the longest prefix of  $w$  of length at most  $n$ .

There is a convenient solution of the pseudoidentity problem for pseudovarieties of the form  $V * D_n$  [3, Section 10.6], which we proceed to describe. Denote by  $A_k$  the set of all words of length  $k$  in  $A^+$  and by  $A_{\leq k}$  all words of length at most  $k$ . There is a unique continuous mapping  $\Phi_n : \overline{\Omega}_A S \rightarrow (\overline{\Omega}_{A_{n+1}} S)^1$  with the following properties:

- (a)  $\Phi_n(w) = 1$  for every  $w \in A_{\leq n}$ ;
- (b)  $\Phi_n(w) = w$  for  $w \in A_{n+1}$ ;
- (c)  $\Phi_n(uv) = \Phi_n(u)\Phi_n(t_n(u)v) = \Phi_n(u i_n(v))\Phi_n(v)$  for all  $u, v \in \overline{\Omega}_A S$ .

For a word  $w \in A^+$ ,  $\Phi_n(w)$  is the word obtained by reading, from left to right, the successive factors of  $w$  of length  $n+1$ . In case  $n=0$ , this is just the identity mapping. For  $n>0$ , the word  $\Phi_n(w)$  can thus be thought of as describing a path in the de Bruijn graph of  $A$  of order  $n$ , that is an element of the free category on this graph. In general, for  $n>0$ , the pseudoword  $\Phi_n(w)$  can be viewed as an element of the free profinite category on the same graph. Note that, for  $n>0$ , the mapping  $\Phi_n$  is not a homomorphism.

**Theorem 2.1** ([3, Theorem 10.6.12]). *Let  $V$  be a pseudovariety that contains some nontrivial monoid and let  $n>0$ . A pseudoidentity  $u=v$  holds in the pseudovariety  $V * D_n$  if and only if  $i_n(u) = i_n(v)$ ,  $t_n(u) = t_n(v)$ , and  $V$  satisfies the pseudoidentity  $\Phi_n(u) = \Phi_n(v)$ .*

Note that, if  $V$  contains the pseudovariety  $RB$ , then the assumption that  $V$  satisfies  $\Phi_n(u) = \Phi_n(v)$  implies that, either  $u, v \in A_{\leq n}$ , or  $\Phi_n(u)$  and  $\Phi_n(v)$  start and end with the same letters, which automatically guarantees the other two conditions in the theorem, namely  $i_n(u) = i_n(v)$  and  $t_n(u) = t_n(v)$ . By Theorem 2.1, the pseudovariety  $RB$  is contained in  $Sl * D_1$ , and the latter is contained in many of the pseudovarieties in which we are interested in this paper which, moreover, satisfy no nontrivial identities. For this reason, we will usually omit reference to the conditions  $i_n(u) = i_n(v)$  and  $t_n(u) = t_n(v)$  when applying Theorem 2.1. The assumption  $Sl * D_1 \subseteq V$  also gives the inclusion  $Sl \subseteq V$  which implies that  $V$  contains a nontrivial monoid.

Another observation regarding Theorem 2.1, which is formulated below as Lemma 2.2, is that, if  $V * D_n = V$  and  $V$  contains  $Sl$ , then the mapping  $\Phi_n$  induces a function  $\Phi_n^V : \overline{\Omega}_A V \rightarrow (\overline{\Omega}_{A_{n+1}} V)^1$  that also satisfies properties (a)–(c). We clarify some technicalities before we state the lemma formally. First, the equality  $V * D_n = V$  implies  $D_n \subseteq V$  and we may assume that  $A_{\leq n} \subseteq \overline{\Omega}_A V$ . Further, for  $w \in A_{n+1} \subseteq \overline{\Omega}_A S$  and  $u \in \overline{\Omega}_A S$  such that  $w = u$  holds in  $V * D_n = V$ , the pseudoidentity  $\Phi_n(u) = \Phi_n(w)$  also holds in  $V$  by Theorem 2.1. Since  $\Phi_n(w) = w \in A_{n+1}$  and  $D_2 \subseteq V$  we get  $\Phi_n(u) = w$  in  $\overline{\Omega}_{A_{n+1}} S$ , which entails the equality  $u = w$ . Altogether, we may assume that  $A_{\leq n+1}$  is embedded in  $\overline{\Omega}_A V$  and  $\pi^{-1}(w) = \{w\}$  for  $w \in A_{\leq n+1}$  and the natural projection  $\pi : \overline{\Omega}_A S \rightarrow \overline{\Omega}_A V$ .

**Lemma 2.2.** *Let  $n>0$  and consider a pseudovariety  $V$  that contains  $V * D_n$  and  $Sl$ . Then there exists a continuous function  $\Phi_n^V$  such that the following*

diagram commutes, where the vertical arrows are the natural projections:

$$(1) \quad \begin{array}{ccc} \overline{\Omega}_A S & \xrightarrow{\Phi_n} & (\overline{\Omega}_{A_{n+1}} S)^1 \\ \pi \downarrow & & \downarrow \sigma_n \\ \overline{\Omega}_A V & \xrightarrow{\Phi_n^V} & (\overline{\Omega}_{A_{n+1}} V)^1. \end{array}$$

Moreover, the following properties hold:

- (a)  $\Phi_n^V(w) = 1$  for every  $w \in A_{\leq n}$ ;
- (b)  $\Phi_n^V(w) = w$  for  $w \in A_{n+1}$ ;
- (c)  $\Phi_n^V(uv) = \Phi_n^V(u)\Phi_n^V(t_n(u)v) = \Phi_n^V(u i_n(v))\Phi_n^V(v)$  for all  $u, v \in \overline{\Omega}_A V$ .

*Proof.* If  $u, v \in \overline{\Omega}_A S$  are such that  $\pi(u) = \pi(v)$ , then the pseudoidentity  $u = v$  holds in  $V = V * D_n$ . By Theorem 2.1, it follows that so does the pseudoidentity  $\Phi_n(u) = \Phi_n(v)$ , whence the equality  $\sigma_n(\Phi_n(u)) = \sigma_n(\Phi_n(v))$  holds. Thus, there is a function  $\Phi_n^V$  such that the diagram commutes. It is continuous because so are  $\sigma_n$ ,  $\Phi_n$ , and  $\pi$ , and  $\overline{\Omega}_A S$  is compact. The verification of properties (a) and (b) for  $\Phi_n^V$  is immediate, while property (c) follows from the commutativity of the diagram (1) and the fact that  $\pi$  is surjective.  $\square$

Although the function  $\Phi_n^V$  is not a homomorphism, we may prove the following consequence of Theorem 2.1, which states that  $\Phi_n^V$  provides a rather convenient means of encoding  $\overline{\Omega}_A V$  in  $\overline{\Omega}_{A_{n+1}} V$ , which we call the *de Bruijn encoding*.

**Theorem 2.3.** *Let  $n > 0$  and consider a pseudovariety  $V$  that contains  $V * D_n$  and  $\text{Sl}$ . Then the mapping  $\Phi_n^V$  is injective on  $\overline{\Omega}_A V \setminus A_{\leq n}$ . Moreover, for  $u, v \in \overline{\Omega}_A V \setminus A_{\leq n}$  and any of Green's equivalence relations  $\mathcal{K}$ ,  $u$  and  $v$  are  $\mathcal{K}$ -related in  $\overline{\Omega}_A V$  if and only if so are  $\Phi_n^V(u)$  and  $\Phi_n^V(v)$  in  $\overline{\Omega}_{A_{n+1}} V$ .*

*Proof.* Given  $w, z \in \overline{\Omega}_A S \setminus A_{\leq n}$ , since the diagram (1) commutes, the equality  $\Phi_n^V(\pi(w)) = \Phi_n^V(\pi(z))$  is equivalent to the pseudoidentity  $\Phi_n(w) = \Phi_n(z)$  being valid in  $V$ . By Theorem 2.1, this in turn is equivalent to the pseudoidentity  $w = z$  being valid in  $V * D_n = V$ , that is  $\pi(w) = \pi(z)$ . Hence, the restriction of  $\Phi_n^V$  to  $\overline{\Omega}_A S \setminus A_{\leq n}$  is injective.

The statement about Green's equivalence relations is handled similarly for all of them. Consider, for instance the  $\mathcal{R}$ -ordering.

Suppose that  $u \geq_{\mathcal{R}} v$  in  $\overline{\Omega}_A V$ , which means that there is some  $w \in (\overline{\Omega}_A V)^1$  such that  $uw = v$ . Applying  $\Phi_n^V$  and taking into account property (c), we obtain  $\Phi_n^V(v) = \Phi_n^V(u)\Phi_n^V(t_n(u)w)$ , which shows that  $\Phi_n^V(u) \geq_{\mathcal{R}} \Phi_n^V(v)$  in  $(\overline{\Omega}_{A_{n+1}} V)^1$ .

Conversely, suppose that  $\Phi_n^V(u) \geq_{\mathcal{R}} \Phi_n^V(v)$ , that is  $\Phi_n^V(u)t = \Phi_n^V(v)$  for some  $t \in (\overline{\Omega}_{A_{n+1}} V)^1$ . Recall that  $u, v \in \overline{\Omega}_A V \setminus A_{\leq n}$ . Since  $V$  contains  $\text{Sl} * D_1$ , the pseudowords  $\Phi_n^V(u)t$  and  $\Phi_n^V(v)$  must have exactly the same factors of length 2. From the definition of  $\Phi_n^V$ , it follows that all factors of length 2 of  $\Phi_n^V(u)t$  must be of the form  $(ax)(xb)$ , where  $x \in A_n$  and  $a, b \in A$ . Since this is precisely the condition that characterizes membership in the image of the function  $\Phi_n^V$ , it follows that  $t = \Phi_n(t_n(u)w)$  for some  $w \in \overline{\Omega}_A V$ . In view of property (c) of the function  $\Phi_n^V$ , it follows that  $\Phi_n^V(v) = \Phi_n^V(u)t = \Phi_n^V(uw)$ .

Since  $\Phi_n^V$  is injective by the first part of the proof, we deduce that  $uw = v$ , which shows that  $u \geq_{\mathcal{R}} v$ .  $\square$

**2.3. Content and related functions.** Let  $S$  be a topological semigroup. For a subset  $X$  of  $S$ , denote by  $\overline{\langle X \rangle}$  the closed subsemigroup generated by  $X$ . For a finite set  $A$ , we say that  $S$  is *A-generated* if there is a mapping  $\varphi : A \rightarrow S$  such that  $\overline{\langle \varphi(A) \rangle} = S$ . Usually, the generating function  $\varphi$  will be understood from the context and not mentioned explicitly. Moreover, whenever we use a letter  $a \in A$  to represent an element of  $S$ , we really mean the element  $\varphi(a)$ .

We say that the  $A$ -generated profinite semigroup  $S$  has a *content function* if the natural projection  $\overline{\Omega}_A S \rightarrow \overline{\Omega}_A S$  factorizes through the unique extension of  $\varphi$  to a continuous homomorphism  $\hat{\varphi} : \overline{\Omega}_A S \rightarrow S$ . Equivalently, for subsets  $B$  and  $C$  of  $A$ , if  $s \in S$  belongs to both  $\overline{\langle B \rangle}$  and  $\overline{\langle C \rangle}$ , then  $B = C$ . Then, for each  $s \in S$ , the unique subset  $B$  of  $A$  such that  $s \in \overline{\langle B \rangle}$  is denoted  $c(s)$  and is called the *content* of  $s$ .

Suppose that  $S$  has a content function. For  $s \in S$ , we denote by  $0(s)$  the set of all  $t \in S^1$  such that there is a factorization  $s = tas'$  with  $c(s) = c(t) \uplus \{a\}$ . The set of all such  $a \in A$  is also denoted  $\bar{0}(s)$ . Dually, the set of all  $t \in S^1$  such that there is a factorization  $s = s'at$  with  $c(s) = c(t) \uplus \{a\}$  is denoted  $1(s)$ , and  $\bar{1}(s)$  is defined similarly. Following [9, Section 3], we say that  $S$  has *0,  $\bar{0}$ , 1, and  $\bar{1}$  functions* if, respectively, each of the sets  $0(s)$ ,  $\bar{0}(s)$ ,  $1(s)$ , and  $\bar{1}(s)$  is a singleton for every  $s \in S$ . Such singleton sets will be identified with their unique elements. Note that if  $S$  has  $0$  and  $\bar{0}$  functions then, by iterating these functions, we conclude that, for  $s \in S$ , the order in which generators occur in  $s$  for the first time from left to right is well determined, and so are the prefixes determined by those first occurrences.

Let  $f$  be one of the functions content,  $0$ ,  $\bar{0}$ ,  $1$ , or  $\bar{1}$ . We say that a pseudovariety  $V$  has the function  $f$  if so does the semigroup  $\overline{\Omega}_A V$  for every finite alphabet  $A$ .

There are many pseudovarieties  $V$  which have content,  $0$ , and  $\bar{0}$  functions. A sufficient condition is given in [9, Proposition 3.5]: it suffices that the pseudovariety  $V$  contain  $Sl$  and be closed under taking right Rhodes expansions (cut down to generators). In turn, a simple sufficient condition for a pseudovariety  $V$  to be closed under right Rhodes expansions is that  $LZ \circledast V = V$  [35]. Dually,  $V$  is closed under left Rhodes expansions if  $RZ \circledast V = V$ . Obvious sufficient conditions for the conjunction of the conditions  $LZ \circledast V = V$  and  $RZ \circledast V = V$  are that  $RB \circledast V = V$  or  $B \circledast V = V$ . Iterating alternately right and left Rhodes expansions on a finite semigroup  $S$ , the process stops (up to isomorphism) in a finite number of steps. The resulting semigroup is known as the *Birget expansion* of  $S$  [13, 14].

**2.4. Equidivisibility and complexity.** A semigroup  $S$  is said to be *equidivisible* if, whenever  $s, t, u, v$  are elements of  $S$  such that  $st = uv$ , there exists some  $w \in S^1$  such that either  $u = sw$  and  $wv = t$ , or  $uw = s$  and  $v = wt$ . This notion was introduced in [27], as a generalization of free semigroup. A pseudovariety of semigroups  $V$  is also said to be *equidivisible* if  $\overline{\Omega}_A V$  is equidivisible for every finite set  $A$ .



It is easy to show that every equidivisible pseudovariety containing  $\mathbf{Sl}$  has  $0$ ,  $\bar{0}$ ,  $1$ , and  $\bar{1}$  functions.

A sufficient condition for equidivisibility has been explicitly given in [5]. We say that a pseudovariety  $\mathbf{V}$  is *closed under concatenation* if the variety of regular languages corresponding to it according to Eilenberg's correspondence [31] enjoys that property, that is, if  $K$  and  $L$  are languages over the same finite alphabet whose syntactic semigroups belong to  $\mathbf{V}$ , then so does the syntactic semigroup of the language  $KL$ . It is proved in [5, Lemma 4.8] that every pseudovariety closed under concatenation is equidivisible. In particular,  $\mathbf{S}$  is equidivisible.

The *closure under concatenation* of a pseudovariety  $\mathbf{V}$  is the smallest pseudovariety closed under concatenation that contains  $\mathbf{V}$ . It may be described as  $\mathbf{A} \circledast \mathbf{V}$  [43, 16]. Hence, a pseudovariety  $\mathbf{V}$  is closed under concatenation if and only if it satisfies the equation  $\mathbf{A} \circledast \mathbf{V} = \mathbf{V}$ .

For a pseudovariety  $\mathbf{V}$  containing  $\mathbf{N}$ , the property of being closed under concatenation also has a very simple and useful topological formulation. Namely, it is equivalent to the multiplication of  $\bar{\Omega}_A \mathbf{V}$  being an open mapping for every finite set  $A$  [5, Lemma 2.3].

A familiar class of examples of pseudovarieties closed under concatenation is given by the pseudovarieties of the form  $\bar{\mathbf{H}}$ , where  $\mathbf{H}$  is an arbitrary pseudovariety of groups. It is indeed a simple exercise to check that  $\mathbf{A} \circledast \bar{\mathbf{H}} = \bar{\mathbf{H}}$ . Note also that  $\bar{\mathbf{H}} * \mathbf{A} = \bar{\mathbf{H}}$ .

Another example is given by the Krohn-Rhodes complexity pseudovarieties  $\mathbf{C}_n$ , which are extensively studied in [41, Chapter 4]. They are defined recursively by  $\mathbf{C}_0 = \mathbf{A}$  and  $\mathbf{C}_{n+1} = \mathbf{C}_n * \mathbf{G} * \mathbf{A}$ . By [41, Corollary 4.9.4], the equality  $\mathbf{A} \circledast \mathbf{C}_n = \mathbf{C}_n$  holds for every  $n \geq 0$ . Another property of interest for the purposes of this paper is that the complexity pseudovarieties  $\mathbf{C}_n$  are monoidal by [41, Proposition 4.3.14]. Thus, we have the following result, which we state here for later reference.

**Proposition 2.4.** *Let  $\mathbf{H}$  be a pseudovariety of groups,  $n \geq 0$ , and  $\mathbf{V}$  be one of the pseudovarieties  $\bar{\mathbf{H}}$  and  $\mathbf{C}_n$ . Then  $\mathbf{V}$  is monoidal and closed under concatenation, it has content,  $0$ ,  $\bar{0}$ ,  $1$ , and  $\bar{1}$  functions, and the equalities  $\mathbf{V} * \mathbf{D} = \mathbf{D} \circledast \mathbf{V} = \mathbf{K} \circledast \mathbf{V} = \mathbf{V}$  hold.*  $\square$

**2.5. Letter cancellation.** We say that an  $A$ -generated topological semigroup  $S$  is *right letter cancelative* if, for every generator  $a \in A$  and all  $s, t \in S$ , if  $sa = ta$  then  $s = t$ . The pseudovariety  $\mathbf{V}$  is said to be *right letter cancelative* if so is each semigroup  $\bar{\Omega}_A \mathbf{V}$  for every finite alphabet  $A$ . Equivalently, if  $\mathbf{V}$  satisfies the pseudoidentity  $ua = va$  over a finite alphabet  $A$ , where  $a \in A$ , then it also satisfies the pseudoidentity  $u = v$ . The dual notion of right letter cancelative is *left letter cancelative*, whose precise definition for a topological semigroup and for a pseudovariety is left to the reader.

The following result assumes familiarity with Eilenberg's correspondence between pseudovarieties of semigroups and varieties of languages. The proof can be considered an exercise in the theory of profinite semigroups, but is included for the sake of completeness. The reader may wish to recall that the variety of languages  $\mathcal{V}$  corresponding to a pseudovariety of semigroups  $\mathbf{V}$  associates with a finite alphabet  $A$  the set  $\mathcal{V}(A)$  of all  $\mathbf{V}$ -recognizable subsets of  $A^+$ , that is subsets that can be recognized by homomorphisms from  $A^+$

into semigroups from  $\mathbf{V}$ . The proof below uses mainly the fact that the topological space  $\overline{\Omega}_A\mathbf{V}$  is the Stone dual of the Boolean algebra  $\mathcal{V}(A)$  [3, Theorem 3.6.1], a fact that is referred in the proof simply as “Stone duality”. This duality may be expressed as follows, where  $\iota : A^+ \rightarrow \overline{\Omega}_A\mathbf{V}$  is the natural homomorphism: a language  $L \subseteq A^+$  belongs to  $\mathcal{V}(A)$  if and only if  $\overline{\iota(L)}$  is open in  $\overline{\Omega}_A\mathbf{V}$  and  $\iota^{-1}(\overline{\iota(L)}) = L$ ; furthermore, the sets  $\overline{\iota(L)}$  suffice to separate points of  $\overline{\Omega}_A\mathbf{V}$ . Moreover, in case  $\mathbf{V}$  contains  $\mathbf{N}$ , the mapping  $\iota$  is injective, the induced topology on  $A^+$  is discrete, and the condition  $\iota^{-1}(\overline{\iota(L)}) = L$  is superfluous [6, Theorem 2.12]. Furthermore, the clopen sets of the form  $\overline{\iota(L)}$  are sufficient to separate points of  $\overline{\Omega}_A\mathbf{V}$ , and so they generate the topology of  $\overline{\Omega}_A\mathbf{V}$ .

**Proposition 2.5.** *Let  $\mathbf{V}$  be a pseudovariety of semigroups,  $\mathcal{V}$  be the corresponding variety of languages. Then, the following conditions are equivalent:*

- (1) *for every finite alphabet  $A$  and every letter  $a \in A$ ,  $L \in \mathcal{V}(A)$  implies  $La \in \mathcal{V}(A)$ ;*
- (2) *the pseudovariety  $\mathbf{V}$  contains  $\mathbf{D}$  and it is right letter cancelative;*
- (3) *the pseudovariety  $\mathbf{V}$  contains  $\mathbf{RZ}$  and it is right letter cancelative.*

*Proof.* (1)  $\Rightarrow$  (2) Let  $A$  be a finite alphabet and let  $u, v \in \overline{\Omega}_A\mathbf{V}$  and  $a \in A$  be such that  $ua = va$ . Assuming (1), we show that the inequality  $u \neq v$  leads to a contradiction.

Let  $\iota : A^+ \rightarrow \overline{\Omega}_A\mathbf{V}$  be the natural homomorphism. Assuming that  $u \neq v$ , by Stone duality there is a  $\mathbf{V}$ -recognizable language  $L \subseteq A^+$  such that  $\overline{\iota(L)}$  contains  $u$  but not  $v$ . Consider a sequence of words  $(v_n)_n$  from  $A^+$  such that  $\lim \iota(v_n) = v$ . It follows that  $\lim \iota(v_n a) = va = ua$ . By hypothesis, the language  $La$  belongs to  $\mathcal{V}(A)$ , which, by Stone duality, entails that the set  $\overline{\iota(La)}$  is open and  $\iota^{-1}(\overline{\iota(La)}) = La$ . Since  $ua \in \overline{\iota(La)}$ , the words  $v_n a$  must belong to  $La$  for all sufficiently large  $n$ . Hence,  $v_n$  lies in  $L$  for all sufficiently large  $n$ , so that  $v \in \overline{\iota(L)}$ , which contradicts the assumption that  $v \notin \overline{\iota(L)}$ .

Hence,  $\mathbf{V}$  is right letter cancelative. That  $\mathbf{V}$  contains  $\mathbf{D}$  also follows from (1) can be seen by iterating the operations  $L \mapsto La$  on  $A^+$ , since the variety of languages corresponding to  $\mathbf{D}$  consists of all languages that are finite Boolean combinations of languages of the form  $A^*u = A^+u \cup a^{-1}A^+u$  ( $a \in A$ ), with  $u$  a finite word.

(3)  $\Rightarrow$  (1) Suppose that  $\mathbf{V} \supseteq \mathbf{RZ}$  and that  $\mathbf{V}$  is right letter cancelative. Let  $L$  be a language from  $\mathcal{V}(A)$ . We show that the condition  $La \notin \mathcal{V}(A)$  leads to a contradiction. By Stone duality,  $\overline{\iota(L)}$  is an open subset of  $\overline{\Omega}_A\mathbf{V}$  but  $\overline{\iota(La)} = \overline{\iota(L)}a$  is not. The latter condition implies that there is  $u \in \overline{\iota(L)}$  such that  $ua = \lim w_n$  for a sequence  $(w_n)_n$  of words in  $A^+ \setminus La$ . Since  $\mathbf{V}$  contains  $\mathbf{RZ}$ , the function  $t_1$  is well defined on  $\overline{\Omega}_A\mathbf{V}$  and it is continuous. Hence, we may assume that, for all  $n$ , there is a factorization  $w_n = v_n a$ . By compactness of  $\overline{\Omega}_A\mathbf{V}$ , we may further assume that the sequence  $(v_n)_n$  converges to some  $v \in \overline{\Omega}_A\mathbf{V}$ . It follows that  $ua = \lim w_n = \lim v_n a = va$  which, since  $\overline{\Omega}_A\mathbf{V}$  is assumed to be right letter cancelative, yields the equality  $u = v$ . As  $u$  was chosen as an element of the open set  $\overline{\iota(L)}$  and  $\lim v_n = v = u$ , we deduce that  $v_n \in L$ , whence also  $w_n \in La$  for all sufficiently large  $n$ , which contradicts the choice of the sequence  $(w_n)_n$ .  $\square$

The language closure property (1) of Proposition 2.5 is not necessary for right letter cancelativity. For example, using the structure theorem for  $\overline{\Omega}_A J$  [3, Theorem 8.2.8], one may show that  $J$  is right letter cancelative.

In view of Proposition 2.5, an obvious sufficient condition for a pseudovariety to be both left and right cancelative is that the corresponding variety of languages be closed under concatenation. For our purposes, we need an alternative sufficient condition, which can be obtained by taking into account some results from [44].

**Proposition 2.6.** *Let  $V$  be a nontrivial monoidal pseudovariety of semigroups such that  $V * D = V$ . Then  $V$  is both left and right letter cancelative.*

*Proof.* By [44, Corollary 3.3], the equality  $V * LI = V$  holds. Suppose that the syntactic semigroup  $S(L)$  of the language  $L \subseteq A^+$  belongs to  $V$  and let  $a \in A$  be a letter. By [44, Lemma 9.8],  $S(La)$  also belongs to  $V$ . We may therefore apply Proposition 2.5 to deduce that  $V$  is right letter cancelative.

To complete the proof, we show that the dual  $V^\rho$  of  $V$ , which is clearly also nontrivial and monoidal, is again such that  $V^\rho * D = V^\rho$ . Indeed, from the equality  $V * LI = V$  we obtain  $V^\rho = (V * LI)^\rho = V^\rho * LI$ , where the second equality is given by [44, Proposition 4.4].  $\square$

**2.6. Basic factorizations.** In this section, we consider a strengthening of the property of a pseudovariety to have content and 0 and  $\bar{0}$  (or 1 and  $\bar{1}$ ) functions.

By a *left basic factorization* of an element  $s$  of a semigroup  $S$  with a content function  $c$ , we mean a factorization of the form  $s = s_0 a s_1$  with  $s_0, s_1 \in S^1$  such that  $c(s) = c(s_0) \uplus \{a\}$ . In such a factorization, the generator  $a$  is said to be the *marker* and  $s_1$  the *remainder*. We say  $S$  has *unique left basic factorizations* if, for any two left basic factorizations  $s = s_0 a s_1$  and  $s = t_0 b t_1$  of the same element, we have  $s_0 = t_0$ ,  $a = b$ , and  $s_1 = t_1$ . Given a generator  $a$  and an element  $s$  of a semigroup  $S$  with unique left basic factorizations, the first occurrence of  $a$ , from left to right, as a factor of  $s$  can be located by iterated left basic factorization on the left factor until it is found as a marker. The factor that follows it is called the *absolute remainder* of  $a$  in  $s$ . Iterating this procedure on the absolute remainders, one may successively locate first occurrences of the letters of any word  $u = a_1 \cdots a_r$  on the generators for which there is a factorization  $s = s_0 a_1 s_1 \cdots a_r s_r$  of a given element of  $S$ . This is called the *left-greedy occurrence* of  $u$  as a subword in  $s$ , and  $s_r$  is called its *remainder*.

We say that a pseudovariety  $V$  has *unique left basic factorizations* if, for every finite alphabet  $A$ ,  $\overline{\Omega}_A V$  has unique left basic factorizations.

The definition of *right basic factorizations* and the property of having *unique right basic factorizations* are the left-right duals of the above notions.

Combining the unilateral version of [4, Proposition 3.4] with the characterization of pseudovarieties whose corresponding varieties of languages are closed under deterministic product [30], we obtain the following sufficient condition for uniqueness of left basic factorizations at the pseudovariety level and its dual.

**Proposition 2.7.** *Let  $V$  be a monoidal pseudovariety containing  $Sl$ . If  $V$  satisfies the equation  $D \circledast V = V$ , then  $V$  has unique left basic factorizations.*

Dually, if  $V$  satisfies the equation  $K \circledast V = V$ , then  $V$  has unique right basic factorizations.

It can be easily checked that many familiar examples of pseudovarieties  $V$  satisfy the equation  $D \circledast V = V$ . Two families of such examples are registered in the following result.

**Corollary 2.8.** *Let  $H$  be an arbitrary pseudovariety of groups. Then the pseudovarieties  $DO \cap \bar{H}$  and  $DS \cap \bar{H}$  have unique left and right basic factorizations.*  $\square$

Another application of Proposition 2.7 is obtained by invoking Proposition 2.4.

**Corollary 2.9.** *The pseudovarieties  $C_n$  have unique left and right basic factorizations and so do the pseudovarieties  $DS \cap C_n$ .*  $\square$

Since the product of a letter by a language is always deterministic, we also have the following immediate consequence of the results from [30]. Alternatively, one may easily show, by iterating on the left factors left basic factorizations, that a pseudovariety with unique left basic factorizations is left letter cancelative.

**2.7. Some special examples.** The aim of this subsection is to prove some auxiliary results which provide examples of pseudovarieties for which a result in Section 4 applies.

**Proposition 2.10.** *Let  $V$  be a monoidal pseudovariety containing  $Sl$  such that  $V * D = V$ . Then the pseudovariety  $W = DA \circledast V$  is such that  $W * D = W$ .*

*Proof.* By Theorem 2.1, a pseudoidentity  $u = v$  holds in  $W * D_n$  if and only if  $W$  satisfies the pseudoidentity  $\Phi_n(u) = \Phi_n(v)$ . On the other hand, by the Basis Theorem for Mal'cev products [33, Theorem 4.1],  $W$  is defined by the pseudoidentities of the form  $((uv)^\omega u)^2 = (uv)^\omega u$ , where  $u$  and  $v$  are pseudowords such that the pseudoidentities  $u = v = v^2$  hold in  $V$ . Thus, to show that  $W * D_n$  is contained in  $W$ , we assume that the pseudoidentities  $u = v = v^2$  hold in  $V$  and we need to prove that the pseudoidentity

$$(2) \quad \Phi_n(((uv)^\omega u)^2) = \Phi_n((uv)^\omega u)$$

holds in  $W$ . As  $D \subseteq V * D = V$ , we must have  $u, v \in \bar{\Omega}_A S \setminus A^+$ .

Since  $V * D_n = V$ , from Theorem 2.1 we deduce that  $i_n(u) = i_n(v)$ ,  $t_n(u) = t_n(v)$ , and the pseudoidentities

$$(3) \quad \Phi_n(u) = \Phi_n(v) = \Phi_n(v^2)$$

hold in  $V$ . Consider the word  $s = \Phi_n(t_n(u)i_n(u))$ . Taking into account property (c) of the function  $\Phi_n$ , we may express  $\Phi_n(v^2)$  as the product  $\Phi_n(v)s\Phi_n(v)$ . Multiplying on the right all sides of the pseudoidentities  $\Phi_n(u) = \Phi_n(v) = \Phi_n(v^2)$  by  $s$ , we deduce that the pseudowords  $u' = \Phi_n(u)s$  and  $v' = \Phi_n(v)s$  are such that the pseudoidentities  $u' = v' = (v')^2$  hold in  $V$ . Hence, the pseudoidentity  $((u'v')^\omega u')^2 = (u'v')^\omega u'$  holds in  $W$ . Using again property (c) of the function  $\Phi_n$ , we deduce that, for the pseudoidentities

$$(4) \quad \Phi_n(((uv)^\omega u)^2)s = (\Phi_n((uv)^\omega u)s)^2 = \Phi_n((uv)^\omega u)s,$$

the first is valid in every finite semigroup, while the second holds in  $W$ . Since  $LI \circledast W \subseteq DA \circledast W = W$ , we know from Proposition 2.7 that  $W$  is right letter cancelative. Hence, from the fact  $W$  satisfies the pseudoidentities (4), it follows that  $W$  also satisfies the pseudoidentity (2).  $\square$

**Corollary 2.11.** *Let  $V$  be a monoidal pseudovariety of semigroups. Then the pseudovariety  $W = DA \circledast (V * A)$  has the following properties:*

- (i)  $W$  is monoidal;
- (ii)  $W$  is both left and right letter cancelative;
- (iii)  $W * D = W$ ;
- (iv)  $B \circledast W = W$ .

*Proof.* By the remarks at the end of Subsection 2.1, we obtain that both  $V * A$  and  $W$  are monoidal, the latter being property (i). The pseudovariety  $V * A$  certainly contains  $Sl$ , as so does  $A$ . Moreover, as  $LI \subseteq DA$ , property (ii) follows from Proposition 2.7. Since the semidirect product is associative and  $A * D = A$ , we have  $(V * A) * D = V * A$ . Invoking Proposition 2.10, we obtain property (iii). Finally, property (iv) follows from the inclusion  $B \subseteq DA$ .  $\square$

### 3. TRANSLATIONAL REPRESENTATIONS

We introduce in this section certain representations of profinite semigroups given by translational action on the minimum ideal. They are explored in this paper to derive the applications in Sections 8 and 9. It is hoped however that they will eventually also shed light on the structure of relatively free profinite semigroups.

**3.1. The translational hull.** This subsection is partly based on [15, Chapter 4].

For topological spaces  $X$  and  $Y$ , denote by  $C(X, Y)$  the space of all continuous functions  $X \rightarrow Y$ . A net  $(f_i)_i$  in  $C(X, Y)$  is said to *converge continuously* to  $f \in C(X, Y)$  if, for every net  $(x_j)_j$  in  $X$  with limit  $x$ , the net  $(f_i(x_j))_{(i,j)}$  converges to  $f(x)$ . The following lemma relates continuous convergence with convergence in the compact-open topology. It is essentially the same as [15, Lemma 4.1].

**Lemma 3.1.** *Let  $X$  be a locally compact space. Then a net  $(f_i)_i$  converges continuously to  $f$  in  $C(X, Y)$  if and only if it converges to  $f$  in the compact-open topology of  $C(X, Y)$ .*

For a topological space  $X$ , the set  $C(X, X)$  is a monoid under composition, which is denoted  $\mathcal{T}_X^\ell$  or  $\mathcal{T}_X^r$  according to whether functions are taken to act and are composed on the left or on the right, respectively. These function spaces are endowed with the compact-open topology. In case  $X$  is locally compact, it follows from Lemma 3.1 that  $\mathcal{T}_X^\ell$  and  $\mathcal{T}_X^r$  are topological semigroups, in the sense that multiplication is continuous. More generally, the continuity of composition follows from [20, Theorem 3.4.2] which states that, for topological spaces  $X, Y, Z$ , the mapping from  $C(Y, Z) \times C(X, Y)$  to  $C(X, Z)$  given by the formula  $(f, g) \mapsto f \circ g$  is continuous under the assumption that  $Y$  is locally compact.

Let  $S$  be a topological semigroup. A *left translation* of  $S$  is a mapping  $\lambda \in \mathcal{T}_S^\ell$  such that  $\lambda(st) = \lambda(s)t$  for all  $s$  and  $t$  in  $S$ . Dually, a *right translation*

is a mapping  $\rho \in \mathcal{T}_S^r$  such that  $(st)\rho = s(t)\rho$  whenever  $s, t \in S$ . The *inner left translation* of  $S$  determined by an element  $s \in S$  is the mapping  $\lambda_s \in \mathcal{T}_S^\ell$  defined by  $\lambda_s(u) = su$ . The *inner right translation*  $\rho_s$  determined by  $s$  is defined dually.

The mappings  $\lambda \in \mathcal{T}_S^\ell$  and  $\rho \in \mathcal{T}_S^r$  are *linked* if  $s\lambda(t) = (s)\rho t$  for all  $s, t \in S$ . A *bitranslation* of  $S$  is a linked pair  $(\lambda, \rho)$  in which  $\lambda$  is a left translation and  $\rho$  is a right translation. Note that, for  $s \in S$ , the pair  $\omega_s = (\lambda_s, \rho_s)$  is a bitranslation, which is called the *inner bitranslation* determined by  $s$ . Note also that the pair in which both components are the identity function on  $S$  is a bitranslation. The *translational hull*  $\Omega(S)$  of  $S$  consists of all bitranslations of  $S$ . Note that  $\Omega(S)$  is a closed submonoid of the product  $\mathcal{T}_S^\ell \times \mathcal{T}_S^r$ . In particular, if  $S$  is a locally compact semigroup, then  $\Omega(S)$  is a topological monoid. Its topology as a subspace of  $\mathcal{T}_S^\ell \times \mathcal{T}_S^r$  is the compact-open topology. The space  $\Omega(S)$  may also be viewed as a space of continuous functions, namely as a subset of  $C(S, S \times S)$ .

A semigroup  $S$  is *right reductive* if the *canonical mapping*  $S \rightarrow \mathcal{T}_S^\ell$  sending each  $s \in S$  to the inner left translation  $\lambda_s$  is injective. A *left reductive* semigroup is defined dually. The semigroup  $S$  is *reductive* if it is both left and right reductive. We also say that  $S$  is *weakly reductive* if the *canonical mapping*  $S \rightarrow \Omega(S)$  sending each  $s \in S$  to  $\omega_s$  is injective; its image is an ideal of  $\Omega(S)$  [22, Corollary 1.11]. Note that every monoid is reductive.

**Theorem 3.2** ([15, Corollary 4.7 and Theorem 4.9]). *Let  $S$  be a compact reductive semigroup  $S$ . Then the compact-open and pointwise convergence topologies coincide on  $\Omega(S)$  and  $\Omega(S)$  is a compact semigroup.*

Recall that a *profinite semigroup* is a residually finite compact semigroup. Equivalently, it is a compact zero-dimensional semigroup [29]. Since zero-dimensionality is preserved by product [20, Theorem 6.2.14] and inherited by subspaces, we obtain the following result.

**Corollary 3.3.** *If  $S$  is a profinite reductive semigroup, then  $\Omega(S)$  is a profinite semigroup.*  $\square$

Given a locally compact closed ideal  $I$  of a topological semigroup  $S$ , the action of  $S$  both on the left and on the right of  $I$  determines a homomorphism  $\omega^I = (\lambda^I, \rho^I) : S \rightarrow \Omega(I)$ , which we call the  *$I$ -representation* of  $S$ . Where convenient, we may sometimes write  $\omega^I(s) = (\lambda^I(s), \rho^I(s))$  instead of  $\omega_s^I = (\lambda_s^I, \rho_s^I)$  to denote the image of  $s \in S$  under the  $I$ -representation of  $S$ .

**Proposition 3.4.** *For a topological semigroup  $S$  and a locally compact closed ideal  $I$ , the  $I$ -representation of  $S$  is continuous.*

*Proof.* Let  $(s_j)_j$  be a convergent net in  $S$  with limit  $s$ . For every convergent net  $(u_k)_k$  in  $I$  with limit  $u$ , the net  $(s_j u_k)_{j,k}$  converges to  $su$  in  $I$  and so the net  $(\lambda_{s_j}^I(u_k))_{j,k}$  converges to  $\lambda_s^I(u)$  in  $I$ . Similarly, the net  $((u_k)\rho_{s_j}^I)_{j,k}$  converges to  $(u)\rho_s^I$ . By Lemma 3.1, the net  $(\omega_{s_j}^I)_j = ((\lambda_{s_j}^I, \rho_{s_j}^I))_j$  converges in  $\Omega(I)$  to  $\omega_s^I = (\lambda_s^I, \rho_s^I)$ . Hence, the function  $\omega^I$  is continuous.  $\square$

**3.2. Actions on the minimum ideal.** Let  $S$  be a profinite semigroup with a minimum ideal  $K$ . Since  $K$  is generated by any of its elements, it is a closed

ideal. By Proposition 3.4, the  $K$ -representation  $\omega^K = (\lambda^K, \rho^K) : S \rightarrow \Omega(K)$  of  $S$  is a continuous homomorphism. Note, that the restriction of  $\omega^K$  to the ideal  $K$  is faithful, because  $K$  is a completely simple semigroup.

Following [41, Definition 4.6.21], we say that  $S$  is *left mapping (LM)* if the representation  $\lambda^K : S \rightarrow \mathcal{T}_K^\ell$  is faithful. The definition of *right mapping (RM)* profinite semigroup is dual. If  $S$  is both left and right mapping, then  $S$  is said to be *generalized group mapping (GGM)*. A GGM profinite semigroup whose minimum ideal is not aperiodic is also said to be *group mapping (GM)*, but we will not be doing this distinction in this paper. We will also be interested in a weakening of the GGM property which is easier to prove and powerful enough for some applications. We say that  $S$  is *weakly generalized group mapping (WGGM)* if, for all distinct elements  $u, v \in S$ , either  $\lambda^K(u) \neq \lambda^K(v)$  and  $(u)\rho^K \neq (v)\rho^K$ , or both  $u$  and  $v$  belong to  $K$  (and, therefore,  $\omega^K(u) \neq \omega^K(v)$ ).

Since the minimum ideal of a GGM profinite semigroup is a profinite reductive semigroup, taking into account the results of Subsection 3.1, we obtain the following statement.

**Theorem 3.5.** *Let  $S$  be a profinite semigroup with minimum ideal  $K$ .*

- (a) *If  $S$  is WGGM then  $\omega^K : S \rightarrow \Omega(K)$  is an embedding of topological semigroups.*
- (b) *If  $S$  is GGM then  $\Omega(K)$  is a profinite semigroup.* □

Thus, if  $S$  is GGM then  $\Omega(K)$  is a profinite semigroup in which  $S$  embeds as a closed subsemigroup. There is also an embedding of  $S$  in  $\Omega(K)$  under the assumption that  $S$  is WGGM, but then there is no longer any guarantee that the topological semigroup  $\Omega(K)$  is profinite, and it may not even be a compact semigroup as an example in Subsection 3.3 shows.

A pseudovariety of semigroups  $\mathbf{V}$  is *GGM* (respectively *WGGM*) if, for every finite non-singleton set  $A$ , the semigroup  $\overline{\Omega}_A \mathbf{V}$  is a GGM (respectively WGGM) semigroup. Trivially, every pseudovariety of groups is GGM. We also say that a pseudovariety of semigroups  $\mathbf{V}$  is *almost GGM* (respectively *almost WGGM*) if, there are arbitrarily large finite alphabets  $A$  such that  $\overline{\Omega}_A \mathbf{V}$  is GGM (respectively WGGM).

**3.3. The translational hull of a profinite completely simple semigroup.** Since the minimum ideal of a profinite semigroup is a profinite completely simple semigroup, Theorem 3.5 motivates the study of the translational hull of profinite completely simple semigroups. This subsection presents some preliminary observations.

We say that a completely simple semigroup has *torsion* if it is not a rectangular group. By a  $2 \times 2$  *maximal subsemigroup* of a semigroup  $S$  we mean a completely simple subsemigroup with exactly two  $\mathcal{R}$ -classes and two  $\mathcal{L}$ -classes which is the union of  $\mathcal{H}$ -classes of  $S$ . We say that a completely simple semigroup  $S$  has *full torsion* if it is not a single  $\mathcal{R}$ -class nor a single  $\mathcal{L}$ -class and every  $2 \times 2$  maximal subsemigroup has torsion. Note that this condition is equivalent to each of the following properties, where  $e$  and  $f$  are arbitrary idempotents of  $S$ :

- if  $e$  and  $f$  are neither  $\mathcal{R}$  nor  $\mathcal{L}$ -equivalent, then the product  $ef$  is not idempotent;

- if  $ef$  is idempotent, then  $ef \in \{e, f\}$ .

A weaker notion is the following. We say that a completely simple semigroup  $S$  has *plenty of torsion on the left* if, for every pair of distinct  $\mathcal{R}$ -equivalent idempotents  $e$  and  $f$ , there is an idempotent  $g$  from the  $\mathcal{L}$ -class of  $e$  such that  $fg \neq e$ . Note that, if  $S$  has full torsion, then every idempotent  $g \neq e$  in the  $\mathcal{L}$ -class of  $e$  has that property.

It is well known that, for an element  $s$  of a compact semigroup, the closed subsemigroup generated by  $s$  contains a unique idempotent, which we denote  $s^0$ . Note that, by definition, it is the limit of some net of (finite) powers of  $s$ . We also denote by  $s^{-1}$  the inverse of  $ss^0$  in the maximal subgroup of  $S$  containing  $ss^0$ . Thus, we have  $s^0 = ss^{-1} = s^{-1}s$ . In a profinite semigroup, the traditional notation is  $s^\omega$  instead of  $s^0$  and  $s^{\omega-1}$  instead of  $s^{-1}$ , coming from the fact that the  $\omega$ -power was first used in the theory of finite semigroups to represent the  $n!$ -powers for sufficiently large  $n$ .

**Proposition 3.6.** *Let  $S$  be a compact completely simple semigroup. Then the following hold:*

- (a) *if  $u, v \in S$  and  $\lambda_u = \lambda_v$ , then  $\lambda_{u^0} = \lambda_{v^0}$  and  $u$  and  $v$  are  $\mathcal{R}$ -equivalent;*
- (b) *if  $u, v \in S$  are such that  $u^0 = v^0$  and  $\lambda_u = \lambda_v$ , then  $u = v$ ;*
- (c) *the semigroup  $S$  is weakly reductive;*
- (d) *the canonical mapping  $S \rightarrow \mathcal{T}_S^\ell$  is injective if and only if  $S$  has plenty of torsion on the left.*

*Proof.* (a) Let  $u$  and  $v$  be two elements of  $S$  and suppose that  $\lambda_u = \lambda_v$ . We deduce that  $u^n w = v^n w$  for every  $w \in S$  and every positive integer  $n$ . Hence, we have  $\lambda_{u^0} = \lambda_{v^0}$ . Since  $u = uu^0 = vu^0$  and  $v = vv^0 = uv^0$ , it follows that  $u$  and  $v$  are  $\mathcal{R}$ -equivalent.

(b) Under the assumptions, we have  $u = uu^0 = uv^0 = vv^0 = v$ .

(c) Let  $u$  and  $v$  be two elements of  $S$  and suppose that  $(\lambda_u, \rho_u) = (\lambda_v, \rho_v)$ . By (a) and its dual,  $u$  and  $v$  lie in the same maximal subgroup of  $S$ , that is  $u^0 = v^0$ . By (b), it follows that  $u = v$ .

(d) Suppose first that  $S$  has plenty of torsion on the left. Let  $u$  and  $v$  be two elements of  $S$ . Suppose that  $\lambda_u = \lambda_v$ . We claim that  $u = v$ . By (a),  $u^0$  and  $v^0$  are  $\mathcal{R}$ -equivalent idempotents. If  $u^0 \neq v^0$  then, since  $S$  has plenty of torsion, there is an idempotent  $g$  in the  $\mathcal{L}$ -class of  $u^0$  such that  $v^0 g \neq u^0 = u^0 g$ , which contradicts the equality  $\lambda_{u^0} = \lambda_{v^0}$  given by (a). Hence, the equality  $u^0 = v^0$  holds and so  $u = v$  by (b), which proves the claim.

Conversely, assume that the canonical mapping is injective and suppose that  $e$  and  $f$  are two distinct  $\mathcal{R}$ -equivalent idempotents. Then there is some  $w \in S$  such that  $ew \neq fw$ . By Green's Lemma, it follows that  $ewe \neq fwe$ . Hence, we may assume that  $w \mathcal{L} e$ . Let  $t$  be the inverse of  $ew$  in the maximal subgroup  $H$  containing  $e$ . Since  $ew$  and  $fw$  are distinct elements of  $H$ , so are  $e = ewt$  and  $f = fwt$ . Since  $wt \mathcal{H} w$  by Green's Lemma, it suffices to observe that  $wt$  is idempotent. Indeed,  $wt \cdot wt = wte \cdot wt = wte = wt$ .  $\square$

In particular, a profinite completely simple semigroup  $S$  is reductive if and only if it has plenty of torsion both on the left and on the right. By Corollary 3.3,  $\Omega(S)$  is then a profinite semigroup.



In the case of a finite discrete completely 0-simple semigroup, the structure of the translational hull has been described in [24, Chapter 7, Facts 2.14 and 2.15]. In view of the translational representation results of Sections 4–7, it seems worthwhile to carry such results to the case of profinite completely simple semigroups. The analogue of [24, Chapter 7, Fact 2.14] is the following result, which only adds topological considerations. The topology we consider on a Rees matrix semigroup  $\mathcal{M}(A, G, B; P)$  is the product topology on  $A \times G \times B$ .

**Proposition 3.7.** *Let  $S = \mathcal{M}(A, G, B; P)$  be a Rees matrix semigroup where  $A$  and  $B$  are compact zero-dimensional spaces,  $G$  is a profinite group, and  $P : B \times A \rightarrow G$  is a continuous function.*

- (a) *The left translations of  $S$  are the functions of the form  $\lambda(a, g, b) = (\varphi(a), \mu(a)g, b)$ , where  $\varphi \in \mathcal{T}_A^\ell$  and  $\mu : A \rightarrow G$  is a continuous function.*
- (b) *The right translations of  $S$  are the functions of the form  $(a, g, b)\rho = (a, g(b)\nu, (b)\psi)$ , where  $\psi \in \mathcal{T}_B^r$  and  $\nu : B \rightarrow G$  is a continuous function.*
- (c) *If  $\lambda$  is a left translation of  $S$  given by  $(\varphi, \mu)$  and  $\rho$  is a right translation of  $S$  given by  $(\psi, \nu)$ , then the pair  $(\lambda, \rho)$  is linked if and only if the following equation holds for all  $a \in A$  and  $b \in B$ :*

$$(5) \quad (b)\nu P((b)\psi, a) = P(b, \varphi(a)) \mu(a).$$

See also [41, Section 5.5.1] for the connection with linear representations.

An extreme non-reductive case, nevertheless of interest, is that of a rectangular band  $S = A \times B$ , where  $A$  and  $B$  are compact, respectively left-zero and right-zero semigroups. It follows from Proposition 3.7 that  $\Omega(S)$  is isomorphic to the product  $\mathcal{T}_A^\ell \times \mathcal{T}_B^r$ . Suppose for instance that  $A$  is the usual realization of the Cantor set in the real line. Noting that the intersections of  $A$  with all intervals of the forms  $[0, c]$  and  $[c, 1]$  ( $c \in [0, 1] \setminus A$ ) are open, if  $(c_n)_n$  is a sequence in  $[0, 1] \setminus A$  converging to 1, then the sequence of characteristic functions of the subsets  $[c_n, 1] \cap A$  of  $A$ , which belong to  $\mathcal{T}_A^\ell$ , converges pointwise to the characteristic function of the subset  $\{1\}$  of  $A$ , which does not belong to  $\mathcal{T}_A^\ell$ . Since convergence in the compact-open topology implies pointwise convergence, it follows that the topological semigroup  $\mathcal{T}_A^\ell$  is not compact, whence neither is  $\Omega(S)$ . The argument can be easily extended to the case where  $A$  or  $B$  contains a subspace homeomorphic to the Cantor set.

In particular, the previous paragraph shows that the assumption that the semigroup  $S$  is reductive cannot be dropped in the statement of Corollary 3.3. Nevertheless, by Proposition 3.4, a profinite rectangular band  $S$  embeds in  $\Omega(S)$  via the  $S$ -representation  $\omega^S$  by inner bitranslations and it follows from the results of Section 4 that  $\Omega(S)$  may admit some much larger profinite subsemigroups containing  $\omega^S(S)$ .

#### 4. SOME SUFFICIENT CONDITIONS FOR WGGM

In this section, we give some first examples of sufficient conditions for a pseudovariety to be WGGM. Further examples of WGGM pseudovarieties are given in Sections 6 and 7.

The next result gives somewhat mild conditions under which the elements of a relatively free profinite semigroup which do not belong to the minimum ideal  $K$  act faithfully on (the left of)  $K$ .

Suppose that the subsemigroup of  $\overline{\Omega}_A \mathbf{V}$  generated by  $A$  is freely generated by  $A$ . As has already been observed in Subsection 2.5, a simple sufficient condition for this property to hold is that  $\mathbf{V}$  contain  $\mathbf{N}$ . We then identify the subsemigroup of  $\overline{\Omega}_A \mathbf{V}$  generated by  $A$  with  $A^+$  and call its elements *finite words*; all other elements of  $\overline{\Omega}_A \mathbf{V}$  are said to be *infinite*. For each  $w \in \overline{\Omega}_A \mathbf{V}$ , denote by  $F(w)$  the set of finite words that are factors of  $w$ , which we also call the *finite factors* of  $w$ .

Here and in the remainder of the paper, we will also use without further comment the property that, for a pseudovariety  $\mathbf{V}$  containing  $\mathbf{LSI}$ , the finite factors of a product  $xy$ , with  $x, y \in \overline{\Omega}_A \mathbf{V}$  are the finite factors of  $x$ , together with the finite factors of  $y$ , together with the words of the form  $x'y'$ , where  $x'$  is a finite suffix of  $x$  and  $y'$  is a finite prefix of  $y$  [11, Lemma 8.2].

**Proposition 4.1.** *Let  $A$  be a non-singleton finite set and let  $\mathbf{V}$  be a monoidal pseudovariety of semigroups satisfying the following conditions:*

- (i)  $\mathbf{V} * \mathbf{D} = \mathbf{V}$ ;
- (ii) the semigroup  $\overline{\Omega}_A \mathbf{V}$  has content, 0, and  $\bar{0}$  functions.

*Let  $K$  be the minimum ideal of  $\overline{\Omega}_A \mathbf{V}$ . If  $u, v \in \overline{\Omega}_A \mathbf{V}$  are such that  $\lambda^K(u) = \lambda^K(v)$ , then either  $u$  and  $v$  are equal or they both belong to  $K$ .*

*Proof.* Condition (i) implies, in particular, that  $\mathbf{V}$  contains the pseudovariety  $\mathbf{N}$ , so that the free semigroup  $A^+$  can be viewed as a subsemigroup of  $\overline{\Omega}_A \mathbf{V}$ , namely as the subsemigroup generated by  $A$ . Since  $A^+$  is dense in  $\overline{\Omega}_A \mathbf{V}$ , a necessary and sufficient condition for an element  $w$  of  $\overline{\Omega}_A \mathbf{V}$  to belong to  $K$  is that  $F(w) = A^+$ .

Suppose that  $\lambda^K(u) = \lambda^K(v)$  with  $u \neq v$ . If  $u, v \in A^+$  then, for an arbitrary  $w \in K$ , from the equality  $\lambda^K(u) = \lambda^K(v)$  we obtain  $uw = vw$  and thus, in view of the hypothesis (i) and Theorem 2.1, one of  $u$  and  $v$  must be a proper prefix of the other, say  $v = uav'$  for some letter  $a \in A$  and some  $v' \in A^*$ . Then, for  $b \in A \setminus \{a\}$ ,  $ub$  is not a prefix of  $v$  and so we have  $ubw \neq vbw$ , which contradicts the assumption that  $\lambda^K(u) = \lambda^K(v)$ . Hence, at least one of pseudowords  $u$  and  $v$  is infinite.

We claim that  $F(u) = F(v)$ . Suppose that there is a finite word  $s$  that is a factor of  $u$  but not of  $v$ . Let  $n = |s|$ . Since the alphabet  $A$  is not a singleton, we may choose a letter  $a \in A \setminus \{t_1(s)\}$ . There is some letter  $b \in A$  such that the word  $sb$  occurs in the pseudoword  $ua^n$ . Now, choose a letter  $c \in A \setminus \{b\}$ , which again only requires the assumption that  $A$  is not a singleton. For an arbitrary element  $w$  of  $K$ , as  $ua^nscw = va^nscw$ , applying  $\Phi_n^{\mathbf{V}}$  we obtain  $\Phi_n^{\mathbf{V}}(ua^nscw) = \Phi_n^{\mathbf{V}}(va^nscw)$ . Since  $s$  is not a factor of  $v$ , the first occurrence of the “letter”  $sc$  in  $\Phi_n^{\mathbf{V}}(va^nscw)$  must occur in the factor  $\Phi_n^{\mathbf{V}}(t_n(v)a^ns)$ . Moreover, note that the only occurrence of  $s$  as a factor of  $va^ns$  is as its suffix. Indeed, we know that it does not occur in  $va^n$  because it does not occur in  $v$  and the last letter of  $s$  is not  $a$ . Thus, all its occurrences must be found in  $a^ns$ . However, if there is a factorization with  $a^ns = xsy$  with the word  $y$  nonempty, then the number of occurrences of the letter  $t_1(s)$  in  $a^ns$  is the number of its occurrences in  $s$ , whereas in  $xsy$  it occurs at

least that number plus one, as it occurs in  $y$ . In particular, we conclude that the “letter”  $sb$  is not a factor of the prefix of  $\Phi_n^V(va^n scw)$  preceding the first occurrence of the “letter”  $sc$ , while the corresponding property fails for  $\Phi_n^V(ua^n scw)$ , which contradicts the hypothesis (ii) in view of Theorem 2.3 and the hypothesis (i). Hence  $F(u) = F(v)$ .

Next, suppose that  $s \in A^+ \setminus F(u)$ . Choose again  $a \in A \setminus \{t_1(s)\}$  and let  $n = |s|$ . For an arbitrary  $w \in K$ , the equality  $\Phi_{n-1}^V(ua^n sw) = \Phi_{n-1}^V(va^n sw)$  holds. Since, as in the preceding paragraph, the first occurrence of the factor  $s$  on  $ua^n s$  and  $va^n s$  is found precisely in the suffix position, by the hypothesis (ii) we deduce that  $\Phi_{n-1}^V(ua^n s) = \Phi_{n-1}^V(va^n s)$ . In view of the injectivity of the function  $\Phi_{n-1}^V$  on the set  $\overline{\Omega}_A V \setminus A_{\leq n-1}$ , given by Theorem 2.3, we deduce that  $ua^n s = va^n s$ . By Proposition 2.6, it follows that  $u = v$ , in contradiction with our initial assumption. This shows that  $F(u) = F(v) = A^+$  and, therefore, that  $u$  and  $v$  belong to  $K$ , which establishes the proposition.  $\square$

Combining Proposition 4.1 with Proposition 3.6, respectively parts (c) and (d), we obtain the following results.

**Theorem 4.2.** *Let  $A$  be a non-singleton finite set and let  $V$  be a monoidal pseudovariety of semigroups satisfying the following conditions:*

- (i)  $V * D = V$ ;
- (ii) *the semigroup  $\overline{\Omega}_A V$  has content, 0,  $\bar{0}$ , 1, and  $\bar{1}$  functions.*

*Then the semigroup  $\overline{\Omega}_A V$  is WGGM.*  $\square$

**Theorem 4.3.** *Let  $A$  be a non-singleton finite set and let  $V$  be a monoidal pseudovariety of semigroups satisfying the following conditions:*

- (i)  $V * D = V$ ;
- (ii) *the semigroup  $\overline{\Omega}_A V$  has content, 0, and  $\bar{0}$  functions;*
- (iii) *the minimum ideal of  $\overline{\Omega}_A V$  has plenty of torsion on the left.*

*Then the semigroup  $\overline{\Omega}_A V$  is LM.*  $\square$

Note that a pseudovariety  $V$  satisfying conditions (i) and (ii) of Proposition 4.1 must contain the pseudovariety  $K$  by Theorem 2.1, and therefore also  $LI = K \vee D$  (see, for instance, [3, Corollary 6.4.14]). In particular, for a non-singleton finite set  $A$ , the minimum ideal  $K$  of  $\overline{\Omega}_A V$  must be a completely simple semigroup with uncountably many  $\mathcal{R}$  and  $\mathcal{L}$ -classes. Since, as it is well known, the restriction of the natural continuous homomorphism  $\overline{\Omega}_A V \rightarrow \overline{\Omega}_A (V \cap G)$  to every maximal subgroup of  $K$  is onto [41, Lemma 4.6.10], the semigroup  $K$  has only trivial subgroups if and only if  $V \subseteq A$ . In case  $V \subseteq A$ , the pseudovariety  $V$  can only be GGM if it is trivial.

In view of Proposition 2.4, we may apply Theorem 4.2 to obtain the following family of examples of WGGM pseudovarieties. Except for the case of the pseudovariety  $A = \bar{1}$ , this will be improved in Section 5.

**Corollary 4.4.** *For every pseudovariety of groups  $H$ , the pseudovariety  $\bar{H}$  is WGGM.*  $\square$

Further examples of WGGM pseudovarieties can be obtained by combining Theorem 4.2 with Corollary 2.11.

**Corollary 4.5.** *If  $\mathbf{V}$  is a monoidal pseudovariety of semigroups, then the pseudovariety  $\text{DA} \widehat{\otimes} (\mathbf{V} * \mathbf{A})$  is WGGM.*  $\square$

Both Corollary 4.4 and the following result may be viewed as particular cases of Corollary 4.5.

**Corollary 4.6.** *For every pseudovariety of groups  $\mathbf{H}$ , the pseudovarieties  $\mathbf{C}_n \cap \bar{\mathbf{H}}$  are WGGM.*  $\square$

The families of examples in Corollaries 4.4 and 4.6, along with many other examples, can also be obtained by applying the next theorem.

**Theorem 4.7.** *Let  $\mathbf{V}$  be an equidivisible pseudovariety containing LSI. Then  $\mathbf{V}$  is WGGM.*

*Proof.* The proof is similar to that of Proposition 4.1. Let  $A$  be a non-singleton alphabet. We consider distinct elements  $u$  and  $v$  of  $\overline{\Omega}_A \mathbf{V}$ , not both in the minimum ideal  $K$ , and assume that  $\lambda(u) = \lambda(v)$ . Since membership in  $K$  is characterized by having all finite words as factors, there is some word  $s \in A^+$  that is not a factor of at least one of  $u$  and  $v$ . Without loss of generality, we may as well assume that  $s \notin F(v)$ . Moreover, since every word containing  $s$  as factor also has the same property, we may replace  $s$  by  $bsba^{|s|+2}$ , where  $a$  and  $b$  are distinct letters from  $A$ , thereby guaranteeing the additional property that  $s$  has no nontrivial overlap with itself. For the remainder of the proof,  $w$  denotes an arbitrary element of  $K$ .

Suppose first that  $s$  is also not a factor of  $u$ . Since  $\lambda(u) = \lambda(v)$ , we deduce that  $usw = vsw$ . By equidivisibility, the  $s$  on the left must match that on the right, so that  $u = v$ , in contradiction with the initial assumption. Hence,  $s \in F(u)$  and we may assume that  $u \in K$ .

Since  $\mathbf{V}$  contains LSI, the ideal  $(\overline{\Omega}_A \mathbf{V})^1 s (\overline{\Omega}_A \mathbf{V})^1 = \overline{A^* s A^*}$  is a clopen subset of  $\overline{\Omega}_A \mathbf{V}$ . Taking also into account that  $(\overline{\Omega}_A \mathbf{V})^1$  is compact, it follows that there are convergent sequences of words  $(x_n)_n$  and  $(y_n)_n$  such that  $u = \lim x_n s y_n$  and  $s$  is not a factor of  $x_n$ . Let  $x = \lim x_n$  and  $y = \lim y_n$ . Since  $u \in K$ , we must have  $y \neq 1$ . Choose  $c \in A \setminus \{i_1(y)\}$ . From the equality  $\lambda(u) = \lambda(v)$ , we obtain  $xs y s c w = u s c w = v s c w$ . By equidivisibility, the first indicated occurrences of  $s$  in  $xs y s c w$  and  $v s c w$  must match, and thus  $c$  should be the first letter of  $y$ , which it is not. This contradiction completes the proof of the theorem.  $\square$

Taking into account the discussion in Subsection 2.4, we deduce the following result.

**Corollary 4.8.** *Every pseudovariety closed under concatenation is WGGM.*  $\square$

Since the pseudovarieties of the form  $\bar{\mathbf{H}}$  and  $\mathbf{C}_n$  are closed under concatenation, Corollaries 4.4 and 4.6 may also be obtained as particular cases of Corollary 4.8.

## 5. TORSION

Our goal is to prove that certain important pseudovarieties of semigroups are GGM. For this purpose, we want to apply Theorem 4.3. While property (i) of Theorem 4.3 is already formulated in terms of closure conditions

on the pseudovariety  $\mathbf{V}$ , properties (ii) and (iii) are structural properties of the semigroup  $\overline{\Omega}_A \mathbf{V}$ , which renders the application of Theorem 4.3 difficult. For property (ii), this is not so serious, since we have already indicated mild conditions that imply it. We proceed to establish sufficient conditions for property (iii) to hold, which will allow us to show that many pseudovarieties of interest satisfy it.

A basic tool to achieve our aim is the semigroup construction presented in Subsection 5.1, which in the school of John Rhodes is known as the *synthesis* construction, which is used as a tool to build arbitrary (finite or infinite) semigroups essentially from groups [39, 38]. The name refers to the synthesis of the Rees matrix semigroup construction with the Krohn-Rhodes Prime Decomposition Theory. In the *synthesis* theory, the top component in our construction (the semigroup  $S$ ) is taken to be a group and the other component (the semigroup  $T$ ) grows successively by the iteration of the construction. In the present paper, it is rather that other component which plays a special role, being taken from an atom in the lattice of pseudovarieties of semigroups, while the top component may be chosen arbitrarily in the pseudovariety. This construction has recently been used in [8], also in connection with irreducibility properties, and in [17] in a rather different context.

**5.1. A semigroup construction.** We follow closely the introduction of the construction given in [8]. Let  $S$  and  $T$  be semigroups and let  $f : S^1 \rightarrow T^1$  be an arbitrary function. The set

$$M(S, T, f) = S \uplus S^1 \times T^1 \times S^1$$

is a semigroup for the multiplication defined by the following formulas for all  $s, s' \in S$ ,  $s_i, s'_i \in S^1$ ,  $t, t' \in T^1$ :

$$\begin{aligned} s \cdot s' &= ss' \\ s \cdot (s_1, t, s_2) &= (ss_1, t, s_2) \\ (s_1, t, s_2) \cdot s &= (s_1, t, s_2s) \\ (s_1, t, s_2) \cdot (s'_1, t', s'_2) &= (s_1, tf(s_2s'_1)t', s'_2). \end{aligned}$$

Given two pseudovarieties of semigroups  $\mathbf{U}$  and  $\mathbf{V}$ , we denote by  $\mathbf{U} \bullet \mathbf{V}$  the pseudovariety generated by all semigroups of the form  $M(S, T, f)$ , with  $S \in \mathbf{U}$  and  $T \in \mathbf{V}$ . As  $S$  is a subsemigroup of  $M(S, T, f)$ ,  $\mathbf{U}$  is contained in  $\mathbf{U} \bullet \mathbf{V}$ . On the other hand, taking  $S = \{1\}$  and  $f(1) = 1 \in T^1$ , we obtain a semigroup  $M(S, T, f)$  whose subsemigroup  $S^1 \times T \times S^1$  is isomorphic with  $T$ , whence  $\mathbf{V}$  is also contained in  $\mathbf{U} \bullet \mathbf{V}$ .

As observed in [8, Lemma 3.1], all subgroups of  $M(S, T, f)$  are isomorphic to subgroups of either  $S$  or  $T$ . The following is an immediate corollary of this observation.

**Corollary 5.1.** *The equation  $\bar{\mathbf{H}} \bullet \bar{\mathbf{H}} = \bar{\mathbf{H}}$  holds for every pseudovariety of groups  $\mathbf{H}$ .  $\square$*

Here are a few other simple yet useful observations.

**Proposition 5.2.** *Let  $\mathbf{H}$  be a pseudovariety of groups. If  $\mathbf{V}$  is a pseudovariety contained in  $\mathbf{DS} \cap \bar{\mathbf{H}}$ , respectively  $\mathbf{CR} \cap \bar{\mathbf{H}}$ , then so is  $\mathbf{V} \bullet \mathbf{H}$ .*

*Proof.* When  $G \in \mathbf{H}$  and  $f : S^1 \rightarrow G$ , the construction  $M(S, G, f)$  gives a semigroup which is the disjoint union of  $S$  with a completely simple semigroup with maximal subgroups isomorphic to  $G$ .  $\square$

Consider the Rees matrix semigroup  $K_p = \mathcal{M}(I, \mathbb{Z}/p\mathbb{Z}, I, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$ , where  $I$  stands for the set  $\{0, 1\}$ ,  $p$  is an arbitrary prime, and we adopt additive notation for the group  $\mathbb{Z}/p\mathbb{Z}$ . Note that  $K_p$  is generated by the idempotents  $(0, 0, 1)$  and  $(1, 0, 0)$ .

**Lemma 5.3.** *Suppose that  $\mathbf{V}$  is a pseudovariety of semigroups satisfying the condition  $\mathbf{V} \bullet \mathbf{Ab}_p = \mathbf{V}$  for a prime  $p$ . Then  $\mathbf{V}$  contains the monoid  $K_p^1$ .*

*Proof.* Let  $G = \mathbb{Z}/p\mathbb{Z}$ . Then,  $G$  belongs to  $\mathbf{Ab}_p$  and, therefore, also to  $\mathbf{V}$ . In case  $p \neq 2$ , consider the mapping  $g : G \rightarrow G$  that sends 2 to 1 and every other element to 0. Then the subsemigroup  $\{0\} \cup \{0, 1\} \times G \times \{0, 1\}$  of  $M(G, G, g)$  is the usual representation of the Rees matrix semigroup  $K_p$  with an identity element adjoined, which shows that  $K_p^1 \in \mathbf{V}$ . In case  $p = 2$ , we consider the mapping  $h : G \times G \rightarrow G$  which sends  $(1, 1)$  to 1 and every other element to 0. Then, it is easily verified that the subsemigroup  $\{(0, 0)\} \cup \{(0, 0), (1, 0)\} \times G \times \{(0, 0), (0, 1)\}$  of  $M(G \times G, G, h)$  is isomorphic with the monoid  $K_2^1$ .  $\square$

**5.2. Some combinatorial lemmas.** Before stating and proving the result that accomplishes the requirement of plenty of torsion in the minimum ideal of Theorem 4.3, we prove some auxiliary combinatorial results.

**Lemma 5.4.** *Let  $A$  be a non-singleton finite alphabet and let  $\mathbf{V}$  be a pseudovariety containing  $\mathbf{LSI}$ . Let  $x, y, z \in \overline{\Omega}_A \mathbf{V}$  be arbitrary pseudowords, and suppose that  $s \in A^+$  is a word such that  $t_2(s)$  is a square and  $s$  is not a factor of  $x$ . Then, there exists a word  $r \in A^+$  such that  $xr \notin \{y, z\}$  and the only occurrence of  $s$  as a factor of  $xrs$  is as a suffix.*

*Proof.* Let  $a \in A$  be the letter such that  $t_2(s) = a^2$  and choose a letter  $b \in A \setminus \{a\}$ . Consider the words  $r_1 = b(ab)^k$ ,  $r_2 = b(ab)^k b$ , and  $r_3 = b(ab)^k b^2$ , where  $k \geq |s|/2$ . As  $\mathbf{D} \subseteq \mathbf{LSI} \subseteq \mathbf{V}$ , the pseudowords  $xr_i \in \overline{\Omega}_A \mathbf{V}$  ( $i = 1, 2, 3$ ) are distinct, for so are their suffixes of length 3. Hence, at least one of them, say  $xr_i$ , is different from both  $y$  and  $z$ ; let  $r = r_i$ . We claim that the only occurrence of  $s$  as a factor of  $xrs$  is as a suffix.

Suppose that there is an occurrence of  $s$  as a factor of  $xrs$  other than as a suffix. Since  $s$  is not a factor of  $x$ , any occurrence of  $s$  as a factor of  $xrs$  must be obtained as a factor of some  $us$ , where  $u$  is a finite suffix of  $xr$ . We may therefore take such  $u \neq 1$  as short as possible, so that  $s$  is a prefix of  $us$ . In particular, there exists a nonempty word  $v$  such that the equality  $us = sv$  holds in  $A^+$ .

Note that, by construction, the word  $r$  ends with the letter  $b$ . If  $v$  is a letter, then it is the last letter of  $s$ , namely  $a$ . Since  $us$  and  $sv$  have the same number of occurrences of the letter  $a$  and  $|u| = |v|$ , it follows  $u = a$ , which contradicts the fact that  $u$  is a suffix of  $xr$ . Hence,  $v$  has length at least two and, therefore,  $a^2$  is a suffix of  $v$ . Since  $us = sv$ , the words  $us$  and  $sv$  have the same number of occurrences of the factor  $a^2$ . As  $u$  ends with the letter  $b$ , it follows that  $a^2$  is a factor of  $u$ . By the choice of  $r$  and as  $u$  is a suffix of  $xr$ , every occurrence of  $a^2$  in  $u$  must come from  $x$ . Thus, there is

a factorization  $u = x'r$ , where  $x'$  is a suffix of  $x$ . Since  $|u| \geq |r| > |s|$  and  $us = sv$ , we conclude that  $s$  is a prefix of  $u = x'r$ . As every occurrence of  $a^2$  in  $u$  comes from  $x'$ , we deduce that  $s$  is actually a prefix of  $x'$ , whence a factor of  $x$ , in contradiction with the hypothesis of the lemma. This establishes the claim.  $\square$

The next lemma may be viewed as a connectivity property of the de Bruijn graphs on alphabets with at least two letters.

**Lemma 5.5.** *Let  $w \in A^*$  be a word and suppose that  $a, b \in A$  are two distinct letters. Then there is a word  $t \in A^*$  such that the word  $wt$  has only one occurrence of the factor  $aw$ , namely as a suffix, and no occurrence of  $bw$ .*

*Proof.* By counting the number of occurrences of the letter  $b$ , we see that no word of the form  $wa^m$  admits  $bw$  as a factor. Then, for  $m = |w|$ ,  $bw$  is not a factor of  $wa^m w$  but clearly  $aw$  is. Hence, the shortest prefix of  $wa^m w$  that admits  $aw$  as a factor is a word of the required form  $wt$ .  $\square$

**5.3. Torsion accomplished.** We now come to the announced sufficient closure conditions on a pseudovariety  $\mathbf{V}$  for  $\overline{\Omega}_A \mathbf{V}$  to have a minimum ideal with plenty of torsion, provided  $|A| \geq 2$ .

**Theorem 5.6.** *Let  $A$  be a non-singleton finite set and let  $\mathbf{V}$  be a monoidal pseudovariety of semigroups satisfying the following conditions:*

- (i)  $\mathbf{V} * \mathbf{D} = \mathbf{V}$ ;
- (ii) the semigroup  $\overline{\Omega}_A \mathbf{V}$  has content, 0 and  $\bar{0}$  functions;
- (iii)  $\mathbf{V} \bullet \mathbf{Ab}_p = \mathbf{V}$  for some prime  $p$ .

*Then the minimum ideal of  $\overline{\Omega}_A \mathbf{V}$  has plenty of torsion on the left.*

*Proof.* Let  $p$  be a prime verifying condition (iii). Also, let  $K$  be the minimum ideal of  $\overline{\Omega}_A \mathbf{V}$  and let  $u$  and  $v$  be distinct  $\mathcal{R}$ -equivalent idempotents of  $K$ . We claim that  $\lambda^K(u) \neq \lambda^K(v)$ . In view of Proposition 3.6, this is sufficient to establish the theorem.

Suppose first that the following condition holds:

$$(6) \quad t_k(u) = t_k(v) \text{ for every } k \geq 1.$$

Since  $u, v \in K$ , we have  $F(u) = F(v) = A^+$ . Suppose that, for every word  $s \in A^+ \setminus A$ , there are factorizations  $u = u_s s z_s$  and  $v = v_s s z_s$  such that  $s \notin F(t_{|s|-1}(s) z_s)$ . Note that the set  $A^+$  ordered by Green's relation  $\geq_{\mathcal{J}}$  is upper directed. By compactness of the space  $(\overline{\Omega}_A \mathbf{V})^1$ , the net  $(u_s, v_s, s, z_s)_{s \in A^+}$  admits a convergent subnet, say with limit  $(u', v', r, z)$ . Since multiplication is continuous, we deduce the equalities  $u = u' r z$  and  $v = v' r z$ . By construction,  $F(r) = A^+$ , and so  $r$  belongs to  $K$ , whence so does  $r z$ . Hence, the  $\mathcal{R}$ -equivalent idempotents  $u$  and  $v$  are both  $\mathcal{L}$ -equivalent to  $r z$ , in contradiction with the assumption that they are distinct. Thus, there is some finite word  $s \in A^+$  of length at least 2 such that, for all factorizations  $u = u_1 s u_2$  and  $v = v_1 s v_2$  where  $s$  is a factor of neither  $t_n(s) u_2$  nor  $t_n(s) v_2$ , with  $n = |s| - 1$ , the pseudowords  $u_2$  and  $v_2$  are infinite and  $t_n(s) u_2 \neq t_n(s) v_2$ , which is equivalent to  $u_2 \neq v_2$  by Proposition 2.6. Note that, if  $s$  has this property, then so does every word of which  $s$  is a factor. Hence, we may assume without loss of generality that  $s$  has the form

$s = bs'ba^{\ell+2}$  where  $a$  and  $b$  are distinct letters from  $A$ ,  $s' \in A^+$ , and  $\ell = |s'|$ . The advantage of such a choice is that  $s$  does not overlap with itself, which makes it easier to locate occurrences of  $s$  in a pseudoword, and  $s$  ends with the square of a letter, which allows us to invoke Lemma 5.4. To simplify the notation, we let  $\bar{s} = t_n(s)$  and  $\bar{u} = t_n(u)$ . Also consider the word  $s_0 = i_n(s)$ .

Write  $u = u_0su_3$  and  $v = v_0sv_3$ , where  $s \notin F(u_0) \cup F(v_0)$ . Since  $u$  and  $v$  are  $\mathcal{R}$ -equivalent elements in  $K$ , there is  $w \in (\overline{\Omega_A V})^1$  such that  $v = uw$ . By Theorem 2.1 and condition (i), we know that  $\Phi_n^V(v) = \Phi_n^V(uw)$ . Since  $s$  has no overlap with itself and it is not a factor of either  $u_0$  or  $v_0$ , by condition (ii) the first occurrences of  $s$ , as a letter from  $A_n$ , in  $\Phi_n^V(u)$  and  $\Phi_n^V(v)$  from left to right as well as the prefixes that determine them must be the same. Hence, we must have  $\Phi_n^V(u_0s_0) = \Phi_n^V(v_0s_0)$  and, therefore, also  $u_0s_0 = v_0s_0$ . Taking into account that  $u$  and  $v$  are idempotents, we obtain the following equalities:

$$(7) \quad u = (u_0su_3u_1su_2)^\omega \quad \text{and} \quad v = (u_0sv_3v_1sv_2)^\omega.$$

We are interested in counting, modulo  $p$ , occurrences of pseudowords of the form  $sts$  in  $uw$  and  $vw$ , where  $s$  does not occur in  $t$  and  $w \in K$  remains to be chosen. The occurrences of such factors in the sections  $su_3u_1s$  and  $sv_3v_1s$  of the expressions (7), pose no problem because of the exponents  $\omega$ . Since  $s$  cannot be found as a factor of any of  $u_0, u_2, v_2$ , what we have to worry about is the possible occurrence of  $s$  as a product  $s_1s_2$  with  $u_0 = s_2u_4$ ,  $u_2 = u_5s_1$ , and similarly for  $v$ . As  $s$  does not overlap with itself, there can be at most one such factorization and, in view of (6), there is one coming from  $u$  if and only if the similar factorization comes from  $v$ . In this case, we have factorizations  $u_0 = s_2u_4$ ,  $u_2 = u_5s_1$ , and  $v_2 = v_5s_1$ . If such a case does not occur, then we take  $u_5 = u_2u_0$  and  $v_5 = v_2u_0$ .

By Lemma 5.4, there exists a finite word  $r$  such that  $\bar{s}u_2r \notin \{\bar{s}u_5, \bar{s}v_5\}$  and  $s$  is not a factor of  $\bar{s}u_2r$ . Since  $t_n(u_2) = t_n(u) = t_n(v) = t_n(v_2)$  by (6), we also know that  $s$  is not a factor of  $\bar{s}v_2r$ . By Proposition 2.6, we obtain

$$(8) \quad \bar{s}u_2rs_0 \notin \{\bar{s}v_2rs_0, \bar{s}u_5s_0, \bar{s}v_5s_0\}.$$

As  $\Phi_n^V$  is injective on the set  $\overline{\Omega_A V} \setminus A_{\leq n}$  by Theorem 2.3, the non-membership condition (8) is preserved after applying this function. Hence, as  $s$  is not a factor of any of the pseudowords in (8), there is some semigroup  $S$  from  $\mathbf{V}$  and some continuous homomorphism  $\varphi : \overline{\Omega_{A_{n+1} \setminus \{s\}} V} \rightarrow S$  such that the following condition holds:

$$(9) \quad \varphi(\Phi_n^V(\bar{s}u_2rs_0)) \notin \{\varphi(\Phi_n^V(\bar{s}v_2rs_0)), \varphi(\Phi_n^V(\bar{s}u_5s_0)), \varphi(\Phi_n^V(\bar{s}v_5s_0))\}.$$

Consider the additive group  $G = \mathbb{Z}/p\mathbb{Z}$  and the semigroup  $M(S, G, \xi)$ , where  $\xi : S^1 \rightarrow G$  maps  $\varphi(\Phi_n^V(\bar{s}u_2rs_0))$  to the generator 1 and every other element to the idempotent 0. Since  $p$  verifies condition (iii), the semigroup  $M(S, G, \xi)$  belongs to  $\mathbf{V}$ . We may therefore extend  $\varphi$  to a continuous homomorphism  $\psi : \overline{\Omega_{A_{n+1}}} V \rightarrow M(S, G, \xi)$  by letting

$$\psi(\alpha) = \begin{cases} (1, 0, 1) & \text{if } \alpha = s, \\ \varphi(\alpha) & \text{if } \alpha \in A_{n+1} \setminus \{s\}. \end{cases}$$



We claim that

$$(10) \quad \psi(\Phi_n^V(urs)) \neq \psi(\Phi_n^V(vrs)).$$

Let us first look at the consequences of this claim, postponing its proof until the next paragraph. Since the two sides of the inequality (10) fall in the same subgroup of the minimum ideal of  $M(S, G, \xi)$ , Green's Lemma implies that  $\psi(\Phi_n^V(ursw)) \neq \psi(\Phi_n^V(vrsw))$  for every  $w \in \overline{\Omega}_A \mathbf{V}$ , in particular for  $w \in K$ . Hence, we have  $\Phi_n^V(ursw) \neq \Phi_n^V(vrsw)$ , whence  $ursw \neq vrsw$ . Thus, the claim yields the inequality  $\lambda^K(u) \neq \lambda^K(v)$  under the assumption that condition (6) holds.

To prove the claim (10), we use the expressions (7) for  $u$  and  $v$ . Taking into account the definition of  $\psi$  and how the multiplication in  $M(S, G, \xi)$  is defined, we may then compute

$$\begin{aligned} \psi(\Phi_n^V(urs)) &= \psi(\Phi_n^V((u_0 s u_3 u_1 s u_2)^\omega r s)) \\ &= \psi\left(\Phi_n^V(u_0 s_0) s (\Phi_n^V(\bar{s} u_3 u_1 s_0) s \Phi_n^V(\bar{s} u_2 u_0 s_0) s)^{\omega-1} \right. \\ &\quad \left. \Phi_n^V(\bar{s} u_3 u_1 s_0) s \Phi_n^V(\bar{s} u_2 r s_0) s \right) \\ &= (\varphi(\Phi_n^V(u_0 s_0)), g + 1, 1), \end{aligned}$$

where

$$g = \begin{cases} -\xi(\varphi(\Phi_n^V(\bar{s} u_4 s_0))) & \text{if } s \in F(u_2 u_0) \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we obtain  $\psi(\Phi_n^V(vrs)) = (\varphi(\Phi_n^V(u_0 s_0)), g, 1)$ , where  $g$  is also given by the above formula. Hence, we have  $\psi(\Phi_n^V(urs)) \neq \psi(\Phi_n^V(vrs))$ , as was claimed.

It remains to treat the cases where (6) fails. Let  $k$  be minimum such that  $t_k(u) \neq t_k(v)$  and let  $n = k - 1$ ,  $s = t_n(u) = t_n(v)$ ,  $as = t_k(u)$ , and  $bs = t_k(v)$ . In particular,  $a$  and  $b$  are distinct letters from  $A$ . By Lemma 5.5, there are words  $r, t \in A^*$  such that  $sr$  has only one occurrence of  $as$  as a factor, namely as a suffix, and none of  $bs$ , and  $st$  has only one occurrence of  $bs$  as a factor, namely as a suffix, and none of  $as$ .

By Theorem 2.3, the elements  $u' = \Phi_n^V(u)$  and  $v' = \Phi_n^V(v)$  of  $\overline{\Omega}_{A_k} \mathbf{V}$  are  $\mathcal{R}$ -equivalent but not  $\mathcal{L}$ -equivalent. Since the monoid  $K_p^1$  belongs to  $\mathbf{V}$  by Lemma 5.3, there is a continuous homomorphism  $\varphi : \overline{\Omega}_{A_k} \mathbf{V} \rightarrow K_p^1$  that maps  $as$  to  $(0, 0, 1)$ ,  $bs$  to  $(1, 0, 0)$ , and every other element of  $A_k$  to the identity element 1. Since the pseudowords  $u'$  and  $v'$  end respectively with the letters  $as$  and  $bs$ , their images  $\varphi(u')$  and  $\varphi(v')$  must be  $\mathcal{R}$ -equivalent but not  $\mathcal{L}$ -equivalent. More precisely, there exist  $i \in \{0, 1\}$  and  $g, h \in \mathbb{Z}/p\mathbb{Z}$  such that  $\varphi(u') = (i, g, 1)$  and  $\varphi(v') = (i, h, 0)$ . If  $g = h$ , then we obtain

$$\begin{aligned} \varphi(\Phi_n^V(ut)) &= \varphi(\Phi_n^V(u) \Phi_n^V(st)) = (i, g, 1)(1, 0, 0) = (i, g + 1, 0), \\ \varphi(\Phi_n^V(vt)) &= \varphi(\Phi_n^V(v) \Phi_n^V(st)) = (i, g, 0)(1, 0, 0) = (i, g, 0). \end{aligned}$$

Similarly, in case  $g \neq h$ , we may calculate

$$\varphi(\Phi_n^V(ur)) = (i, g, 1) \quad \text{and} \quad \varphi(\Phi_n^V(vr)) = (i, h, 1).$$

In both cases, by Green's Lemma, we may then take any  $w \in K$  to deduce that either  $\varphi(\Phi_n^V(utw)) \neq \varphi(\Phi_n^V(vtw))$  or  $\varphi(\Phi_n^V(urw)) \neq \varphi(\Phi_n^V(vrw))$ . This

shows that  $utw \neq vtw$  or  $urw \neq vrw$  and, therefore,  $\lambda^K(u) \neq \lambda^K(v)$  which concludes the proof of the theorem.  $\square$

Combining Theorems 4.3 and 5.6, we obtain the following result, which allows us to show that many pseudovarieties of interest are GGM.

**Theorem 5.7.** *Let  $\mathbf{V}$  be a monoidal pseudovariety of semigroups satisfying the following conditions:*

- (i)  $\mathbf{V} * \mathbf{D} = \mathbf{V}$ ;
- (ii) *the pseudovariety  $\mathbf{V}$  contains  $\mathbf{Sl}$  and it is closed under Birget expansions;*
- (iii)  $\mathbf{V} \bullet \mathbf{Ab}_p = \mathbf{V}$  for some prime  $p$ .

*Then the pseudovariety  $\mathbf{V}$  is GGM.*  $\square$

Taking into account Corollary 5.1, we may apply Theorem 5.7 to many familiar pseudovarieties, thus improving Corollary 4.4.

**Corollary 5.8.** *For every nontrivial pseudovariety of groups  $\mathbf{H}$ , the pseudovariety  $\bar{\mathbf{H}}$  is GGM.*  $\square$

## 6. WGGM FOR SUBPSEUDOVARIETIES OF $\mathbf{DS}$

Many pseudovarieties of interest are contained in the pseudovariety  $\mathbf{DS}$ . Although  $\mathbf{DS}$  can be easily seen to be closed under Birget expansions and  $\mathbf{DS} \bullet \mathbf{G} = \mathbf{DS}$  by Proposition 5.2,  $\mathbf{DS} * \mathbf{D} \neq \mathbf{DS}$  since, for instance,  $\mathbf{Sl} * \mathbf{D}$  contains the aperiodic five-element Brandt semigroup  $B_2$  while  $\mathbf{DS}$  is precisely the largest pseudovariety that does not contain  $B_2$ . Thus, to establish that suitable subpseudovarieties of  $\mathbf{DS}$  are GGM, we have to develop an alternative approach. In fact, we only manage to prove WGGM. The basic idea is that, for every pseudovariety  $\mathbf{V}$  in the interval  $[\mathbf{J}, \mathbf{DS}]$ , where  $\mathbf{J}$  is the pseudovariety of all finite  $\mathcal{J}$ -trivial semigroups, membership in the minimum ideal of  $\bar{\Omega}_A \mathbf{V}$  is characterized by the property of admitting all finite words as subwords<sup>1</sup> [3, Theorems 8.1.7 and 8.1.10]. Since the minimum ideal of  $\bar{\Omega}_A \mathbf{J}$  is trivial, we need a larger pseudovariety to be able to start a program mimicking that developed in previous sections. Since  $\mathbf{Sl} \subseteq \mathbf{J}$ , such a semigroup  $\bar{\Omega}_A \mathbf{V}$  has automatically a content function.

In compensation for dropping the hypothesis  $\mathbf{V} * \mathbf{D} = \mathbf{V}$ , we need to reinforce the hypothesis of having 0 and  $\bar{0}$  functions with the stronger condition of uniqueness of left basic factorizations.

**Proposition 6.1.** *Let  $\mathbf{V}$  be a pseudovariety in the interval  $[\mathbf{J}, \mathbf{DS}]$  and let  $A$  be a non-singleton finite alphabet. If  $\mathbf{V}$  has unique left basic factorizations and  $u, v \in \bar{\Omega}_A \mathbf{V}$  are such that  $\lambda^K(u) = \lambda^K(v)$  then either  $u$  and  $v$  are equal or they both belong to the minimum ideal  $K$ .*

*Proof.* Suppose that there is some finite subword of  $u$  that is not a subword of  $v$ . Choose  $za$  to be such a word of minimum length, with  $a \in A$ , so that  $za$  is a subword of  $u$  but not of  $v$ , while  $z$  is a subword of  $v$ . Since  $|A| \geq 2$ , we may choose a letter  $b \in A$  which is different from the first letter of the remainder in the left-greedy occurrence of  $za$  as a subword in  $ua$ .

<sup>1</sup> For a pseudoword  $w \in \bar{\Omega}_A \mathbf{V}$  and a finite word  $s \in A^+$ , we say that  $s$  is a *subword* of  $w$  if there are factorizations  $s = s_1 \cdots s_n$  and  $w = w_0 s_1 w_1 \cdots s_n w_n$ , where  $s_1, \dots, s_n \in A$  and  $w_0, \dots, w_n \in (\bar{\Omega}_A \mathbf{V})^1$ .

Let  $w$  be any element of the minimum ideal  $K$  of  $\overline{\Omega}_A \mathbf{V}$ . Since  $za$  is not a subword of  $v$ , but  $z$  is, the remainder of the left greedy occurrence of  $za$  in  $vabw$  is  $bw$ . On the other hand, since  $za$  is a subword of  $u$  and by the choice of the letter  $b$ , the remainder of the left greedy occurrence of  $za$  in  $uabw$  starts with a letter different from  $b$ . In view of uniqueness of left basic factorizations in  $\overline{\Omega}_A \mathbf{V}$ , it follows that  $uabw \neq vabw$ , which contradicts the assumption that  $\lambda^K(u) = \lambda^K(v)$ . Hence  $u$  and  $v$  have the same finite subwords.

Suppose that there is some finite word that is not a subword of  $u$ . Let  $za$  be such a word of minimum length, with  $a \in A$ . Let  $w$  be an arbitrary element of  $K$ . From the hypothesis that  $\lambda^K(u) = \lambda^K(v)$ , we deduce that  $uaw = vaw$ . But, since  $z$  is a subword of both  $u$  and  $v$ , while  $za$  is not, in the left greedy occurrence of  $za$  in  $uaw$  and  $vaw$ , the indicated occurrences of  $a$  must be the chosen occurrences of the last letter of  $za$ . Hence, we must have  $u = v$ . Thus, if  $u \neq v$ , then  $u$  and  $v$  both admit all finite words as subwords and so they belong to  $K$ .  $\square$

Combining Proposition 6.1 with its dual and Proposition 3.6(c), we deduce the following result.

**Theorem 6.2.** *Let  $\mathbf{V}$  be a pseudovariety in the interval  $[\mathbf{J}, \mathbf{DS}]$  that has unique left and right basic factorizations. Then  $\mathbf{V}$  is WGGM.*  $\square$

Combining Theorem 6.2 and Corollary 2.8, we obtain the following important examples of WGGM pseudovarieties.

**Corollary 6.3.** *For an arbitrary pseudovariety of groups  $\mathbf{H}$ , the pseudovarieties  $\mathbf{DO} \cap \bar{\mathbf{H}}$  and  $\mathbf{DS} \cap \bar{\mathbf{H}}$  are WGGM.*  $\square$

We conjecture that every pseudovariety of the form  $\mathbf{DS} \cap \bar{\mathbf{H}}$ , where  $\mathbf{H}$  is a nontrivial pseudovariety of groups, is actually GGM.

Another interesting class of examples is obtained by combining Theorem 6.2 with Corollary 2.9, which leads to the following result.

**Corollary 6.4.** *The pseudovarieties  $\mathbf{DS} \cap \mathbf{C}_n$  are WGGM.*  $\square$

## 7. GGM FOR SUBPSEUDOVARIETIES OF $\mathbf{CR}$

This section is dedicated to proving GGM or its weakened versions for various subpseudovarieties of  $\mathbf{CR}$ .

**Proposition 7.1.** *Let  $A$  be a non-singleton finite alphabet and let  $\mathbf{V}$  be a subpseudovariety of  $\mathbf{CR}$  such that  $\overline{\Omega}_A \mathbf{V}$  has content, 0, and  $\bar{0}$  functions. Suppose further that at least one of the following conditions holds:*

- (i)  $|A| \geq 3$  and  $\mathbf{V}$  contains  $\mathbf{RZ}$ ;
- (ii) the pseudovariety  $\mathbf{V}$  contains some nontrivial group.

*If  $u, v \in \overline{\Omega}_A \mathbf{V}$  are two distinct elements, then either  $\lambda^K(u) \neq \lambda^K(v)$  or both  $u$  and  $v$  belong to the minimum ideal.*

*Proof.* Recall that  $\mathbf{CR}$  is contained in  $\mathbf{DS}$ . Thus,  $\mathbf{V}$  belongs to the interval  $[\mathbf{SI}, \mathbf{DS}]$ , and so the regular  $\mathcal{J}$ -classes of  $\overline{\Omega}_A \mathbf{V}$  are characterized by the content of their elements [3, Theorem 8.1.7]. Since  $\mathbf{V} \subseteq \mathbf{CR}$ , there are no other  $\mathcal{J}$ -classes. In particular, the minimum ideal  $K$  consists precisely of the elements

of full content. Suppose that  $u, v \in \overline{\Omega}_A \mathbf{V}$  are distinct elements such that  $\lambda^K(u) = \lambda^K(v)$ . Let  $w$  be an arbitrary element of  $K$ .

Suppose that there is some letter  $a \in A \setminus (c(u) \cup c(v))$ . From the equality  $uaw = vaw$ , applying the function 0 sufficiently many times, we obtain  $u = v$ , in contradiction with the hypothesis. Hence, every letter from  $A$  occurs in either  $u$  or  $v$ . Assume that there is a letter  $a$  that occurs in  $u$  but not in  $v$ . Let  $u = u_0 a u_1$  be a factorization of  $u$  such that  $a \notin c(u_0)$ . Then, from the equality  $\lambda^K(u) = \lambda^K(v)$ , we obtain  $u_0 a u_1 a w = u a w = v a w$ , which entails  $u_0 = v$ , since  $a \notin c(v)$ , whence  $c(v) \subseteq c(u)$  and  $c(u) = A$ .

Suppose first that  $\mathbf{V}$  contains some nontrivial group. Let  $p$  be a prime such that  $\mathbf{Ab}_p \subseteq \mathbf{V}$  and consider the natural projection  $\pi : \overline{\Omega}_A \mathbf{V} \rightarrow \overline{\Omega}_A \mathbf{Ab}_p$ , where  $\overline{\Omega}_A \mathbf{Ab}_p \simeq (\mathbb{Z}/p\mathbb{Z})^A$ ; the mapping  $\pi$  counts modulo  $p$  the number of occurrences of each letter. Let  $b$  be a letter from  $c(v)$  and note that  $\lambda^K(u) = \lambda^K(v)$  also yields  $u_0 a u_1 b a w = u b a w = v b a w$ , which now entails  $u_0 = v b$ , since  $a \notin c(v b)$ . Hence  $\pi(v) = \pi(u_0) = \pi(v b)$ , which is absurd since the  $b$ -components of  $\pi(v)$  and  $\pi(v b)$  are distinct.

Consider finally the aperiodic case, where  $\mathbf{RZ} \subseteq \mathbf{V} \subseteq \mathbf{CR} \cap \mathbf{A} = \mathbf{B}$  and  $|A| \geq 3$ . Choose  $b \in A \setminus \{a, t_1(u_0)\}$ . Then, from the equality  $\lambda^K(u) = \lambda^K(v)$ , we obtain  $u_0 a u_1 b a w = u b a w = v b a w$ , whence  $u_0 = v b$ . Hence, we have  $t_1(u_0) = b$ , in contradiction with the choice of  $b$ .  $\square$

Note that  $\lambda^K(ab) = \lambda^K(a)$  in  $\overline{\Omega}_{\{a,b\}} \mathbf{B}$ , which shows that the restriction  $|A| \geq 3$  cannot be dropped from the hypothesis of Proposition 7.1 in the aperiodic case.

Combining Propositions 7.1 and 3.6(c), we obtain the following result.

**Theorem 7.2.** *Let  $\mathbf{V}$  be a pseudovariety in the interval  $[\mathbf{Sl}, \mathbf{CR}]$ , and suppose  $\mathbf{V}$  is closed under Birget expansions. Then  $\mathbf{V}$  is almost WGGM. Moreover, if  $\mathbf{V}$  contains some nontrivial group then  $\mathbf{V}$  is WGGM.*  $\square$

Combining Theorem 7.2 with Proposition 2.4, we obtain the following family of further examples of WGGM pseudovarieties.

**Corollary 7.3.** *For  $n > 0$ , the pseudovarieties  $\mathbf{CR} \cap \mathbf{C}_n$  are WGGM.*  $\square$

Theorem 7.2 also yields that  $\mathbf{B}$  is almost WGGM while, for a nontrivial pseudovariety of groups  $\mathbf{H}$ ,  $\mathbf{CR} \cap \bar{\mathbf{H}}$  is WGGM. The remainder of this section is dedicated to proving that  $\mathbf{CR} \cap \bar{\mathbf{H}}$  is actually GGM, with one exception, in which it is almost GGM.

The pseudoidentity problem for  $\mathbf{CR} \cap \bar{\mathbf{H}}$  has been solved in [10]. The solution is in a sense similar to Theorem 2.1 but involving other parameters. Two pseudowords must have the same content to be equal over  $\mathbf{CR} \cap \bar{\mathbf{H}}$ . The roles of  $i_n$  and  $t_n$  are played by the pairs of functions  $(0, \bar{0})$  and  $(1, \bar{1})$ , while that of the function  $\Phi_n$  is taken by the profinite version of Kaĉourek and Polák's characteristic function [23]. For a word  $w$ , the *characteristic sequence*  $\chi(w)$  is the word that is obtained by reading from left to right the maximal factors that miss exactly one letter from  $w$ . For pseudowords, the definition is technically complicated and is made in terms of a pseudopath in a certain free profinite category over a profinite graph with infinitely many vertices. In fact, this poses in general delicate problems which were overlooked in [10], as observed in [5], namely the free category generated by the graph may not

be dense in the free profinite category over the same graph. However, using the techniques of [5], A. Costa and the first author have been able to show that, due to the special nature of the graph, the approach in [10] works fine as the density condition is fulfilled [7].

The graph in question associated with a pseudovariety of groups  $\mathbf{H}$ , denoted  $\partial_X \mathbf{H}$ , is similar to the de Bruijn graph, being associated with a fixed subset  $X$  of the alphabet  $A$ , containing at least two letters. The edges are the pseudowords  $w$  with content contained in  $X$  and missing just one letter, where two edges are identified if the pseudoidentity they determine is valid in  $\mathbf{CR} \cap \bar{\mathbf{H}}$ . The extremes of such an edge  $w$  are the pseudowords  $0(w)$  and  $1(w)$ , missing exactly two letters from  $X$ . Let  $[X]$  be the set of words in  $A^+$  of content  $X$ . The characteristic sequence can be viewed as a function from  $[X]$  to the set of paths in the graph  $\partial_X \mathbf{H}$ . It turns out that it extends uniquely to a continuous function, which we denote  $\chi_{\mathbf{H}}$ , from  $\overline{[X]}$  to the set of pseudopaths of the same graph. The following result provides a recursive criterion for the validity of pseudoidentities in pseudovarieties of the form  $\mathbf{CR} \cap \bar{\mathbf{H}}$ . The term “recursive” is used here in the sense that the criterion for equality calls itself repeatedly on pseudoidentities involving smaller contents.

**Theorem 7.4** ([10, Theorem 3.9]). *Let  $\mathbf{H}$  be a pseudovariety of groups and let  $u, v \in \overline{\Omega}_A \mathbf{S}$ . Then the pseudovariety  $\mathbf{CR} \cap \bar{\mathbf{H}}$  satisfies the pseudoidentity  $u = v$  if and only if each of the following conditions holds:*

- (i)  $c(u) = c(v)$ ;
- (ii) the pseudoidentity  $0(u) = 0(v)$  holds in  $\mathbf{CR} \cap \bar{\mathbf{H}}$ ;
- (iii) the pseudoidentity  $1(u) = 1(v)$  holds in  $\mathbf{CR} \cap \bar{\mathbf{H}}$ ;
- (iv) either  $|c(u)| = 1$  and the pseudoidentity  $u = v$  holds in  $\mathbf{H}$ , or  $|c(u)| > 1$  and the pseudoidentity  $\chi_{\mathbf{H}}(u) = \chi_{\mathbf{H}}(v)$  holds in  $\mathbf{H}$ .

We are interested in distinguishing elements of  $\overline{\Omega}_A \mathbf{S}$  that, projected in  $\overline{\Omega}_A(\mathbf{CR} \cap \bar{\mathbf{H}})$ , fall in the same subgroup of the minimum ideal, where  $A$  is a non-singleton finite alphabet. For such elements, conditions (i)–(iii) of Theorem 7.4 are automatically fulfilled. Thus, the distinction must be done through the condition of the pseudoidentity  $\chi(u) = \chi(v)$  failing in  $\mathbf{H}$ . We want to do it under minimal assumptions, namely that the pseudovariety  $\mathbf{H}$  is nontrivial, say it contains  $\mathbf{Ab}_p$ , where  $p$  is prime. Indeed, it suffices to show that the (profinite) numbers of occurrences in the two pseudowords in question of some maximal factor of content missing just one letter, modulo equality over  $\mathbf{CR} \cap \bar{\mathbf{H}}$ , can be distinguished modulo  $p$ . Alternatively, we may work directly in  $\overline{\Omega}_A(\mathbf{CR} \cap \bar{\mathbf{H}})$ , which avoids the need to consider the identification over  $\mathbf{CR} \cap \bar{\mathbf{H}}$  of maximal factors missing just one letter.

**Theorem 7.5.** *Let  $\mathbf{H}$  be a nontrivial pseudovariety of groups and let  $A$  be a nonempty finite alphabet. Then the semigroup  $\overline{\Omega}_A(\mathbf{CR} \cap \bar{\mathbf{H}})$  is GGM whenever at least one of the following conditions holds:*

- (i)  $|A| \neq 2$ ;
- (ii)  $\mathbf{H} \neq \mathbf{Ab}_2$ .

*Proof.* The case of a singleton alphabet  $A$  is obvious, since then the semigroup  $\overline{\Omega}_A(\mathbf{CR} \cap \bar{\mathbf{H}})$  is a group. We therefore assume from hereon that  $|A| \geq 2$ .

In view of Theorem 7.2, Proposition 3.6(d), and duality, it remains to show that, under the hypotheses (i) or (ii), the minimum ideal  $K$  of the semi-group  $\overline{\Omega}_A(\mathbf{CR} \cap \mathbf{H})$  has plenty of torsion on the left. So, suppose that  $e, f \in K$  are distinct  $\mathcal{R}$ -equivalent idempotents. Then we have  $0(e) = 0(f)$  and  $\bar{0}(e) = \bar{0}(f)$  while  $1(e) \neq 1(f)$  or  $\bar{1}(e) \neq \bar{1}(f)$ . Note that  $e = (0(e)\bar{0}(e)\bar{1}(e)1(e))^\omega$  and  $f = (0(e)\bar{0}(e)\bar{1}(f)1(f))^\omega$ , because each pair of idempotents in these equalities lie in the same  $\mathcal{H}$ -class. We show that there is an idempotent  $g$  from the  $\mathcal{L}$ -class of  $e$  such that  $fg \neq eg$ , which establishes that  $K$  has plenty of torsion on the left.

Let  $x$  be an arbitrary element of  $\overline{\Omega}_A(\mathbf{CR} \cap \mathbf{H})$ . Note that, if  $xf$  is idempotent but  $xe$  is not then, since they are  $\mathcal{R}$ -equivalent by Green's Lemma, we obtain  $xe(xe)^\omega = xe \neq (xe)^\omega = xf(xe)^\omega$  and so  $g = (xe)^\omega$  has the desired property. Let  $a = \bar{1}(e)$  and let  $u$  be a word with  $c(u) = A \setminus \{a\}$ . Taking  $x = (auf)^\omega$ , for which  $xf$  is idempotent, we conclude that we may assume that  $xe$  is also idempotent. It follows that

$$(11) \quad xe = ((auf)^\omega e)^\omega = (aua1(e))^\omega \text{ and } xf = (auf)^\omega = (au\bar{1}(f)1(f))^\omega.$$

Suppose first that  $\bar{1}(f) = a$ . Then  $1(e)$  and  $1(f)$  must be distinct. We claim that it is possible to choose  $v \in (A \setminus \{a\})^*$  such that, for  $g_v = (va1(e))^\omega$ , we have  $xeg_v \neq xfg_v$ . Since  $xf g_v$  and  $xeg_v = xe$  lie in the same maximal subgroup of  $K$ , to prove that they are distinct it suffices to show that there is some maximal factor  $w$  that misses exactly the letter  $a$  and such that the (profinite) numbers of times it appears in  $xf g_v$  and  $xeg_v$  are not congruent modulo  $p$ . Because of the special form (11) that the pseudowords  $xe$  and  $xf$  have, any two consecutive occurrences of  $a$  in  $xeg_v$  and  $xf g_v$  are separated by pseudowords of content  $A \setminus \{a\}$ . Thus, the factors in question are precisely those that appear between two consecutive occurrences of the letter  $a$ , together with the factor after the last occurrence of  $a$ . Table 1 gives the number of occurrences of such factors, taken in the profinite completion of the additive group  $\mathbb{Z}$ .

	$xeg_v = (aua1(e))^\omega(va1(e))^\omega$	$xf g_v = (aua1(f))^\omega(va1(e))^\omega$
$u$	0	0
$1(e)$	0	1
$1(f)$	0	-1
$1(e)v$	0	-1
$1(f)v$	0	1

TABLE 1.

Table 1 does not take into account possible equalities between some of the elements in the first column, in which case the corresponding remainders of the rows should be summed. The possible equalities with  $u$  may be ignored since the values in the corresponding row sum are the same. If we choose  $v$  to be a letter then we guarantee the inequalities  $1(e)v \neq 1(e)$  and  $1(f)v \neq 1(f)$ . Indeed, if  $1(e)v = 1(e)$  holds in  $\mathbf{CR} \cap \mathbf{H}$  then, in particular,  $1(e)v = 1(e)$  holds in  $\mathbf{H} \subseteq \mathbf{CR} \cap \mathbf{H}$ . This gives, for a letter  $v$ , that  $v = 1$  holds in  $\mathbf{H}$ . Hence  $\mathbf{H} = 1$ , a contradiction.

In case  $|A| = 2$ , so that  $H \neq \text{Ab}_2$ , let  $v = b$  be the only letter in  $A \setminus \{a\}$ . Then, since  $1(e) \neq 1(e)b, 1(f)$ , then the total for  $1(e)$  in the column of Table 1 headed by  $xfg_v$  is either 1 or 2, which is different from the null total corresponding to the other column.

In case  $|A| \geq 3$ , let  $b = t_1(1(e))$ . If  $b$  is also the last letter of  $1(f)$ , and  $c$  is a letter from  $A \setminus \{a, b\}$ , then the three pseudowords  $1(e), 1(f), 1(e)c$  are distinct, where the inequality  $1(f) \neq 1(e)c$  follows from the fact that the two sides end with different letters. Taking  $v = c$ , of the four elements  $1(e), 1(f), 1(e)v, 1(f)v$ , at least one is not equal to any of the others and the corresponding row in Table 1 shows that  $xeg_v \neq xfg_v$ . If  $c = t_1(1(f)) \neq b$ , then similarly the three pseudowords  $1(e), 1(f), 1(f)c$  are distinct, and the same argument applies.

It remains to consider the case where the letter  $b = \bar{1}(f)$  is such that  $b \neq a$ , so that  $a$  occurs in  $1(f)$ . Note that there is a factorization  $1(f) = (1(f))^{\omega+1} = w_0aw_1aw_2$ , where  $c(w_0)$  and  $c(w_2)$  are both contained in  $A \setminus \{a, \bar{1}(f)\}$ . Proceeding as in the preceding case, we obtain Table 2, provided we take  $v$  such that  $c(v) = A \setminus \{a\}$ , where we take into account that the contribution of the factors in question that appear within  $w_1$  is null because they appear  $\omega$  times, while there are none within  $w_2$  because  $a, b \notin c(w_2)$ .

	$xeg_v = (aual(e))^\omega(val(e))^\omega$	$xfg_v = (aubw_0aw_1aw_2)^\omega(val(e))^\omega$
$u$	0	0
$1(e)$	0	1
$1(e)v$	0	-1
$ubw_0$	0	0
$w_2v$	0	1

TABLE 2.

Since the numeric values of the sum of all the rows are distinct in every nontrivial cyclic group, we deduce that  $xeg_v \neq xfg_v$ , which completes the proof of the theorem.  $\square$

Using Theorem 7.4, one may check that the two elements  $(ab)^\omega$  and  $(ab^2)^\omega$  of the minimum ideal of  $\bar{\Omega}_{\{a,b\}}(\text{CR} \cap \bar{\text{Ab}}_2)$  have the same image under  $\lambda^K$ .

The following result is less precise than Theorem 7.5 but sufficient for the applications in Sections 8 and 9.

**Corollary 7.6.** *For a nontrivial pseudovariety of groups  $H$ , the pseudovariety  $\text{CR} \cap \bar{H}$  is GGM, unless  $H = \text{Ab}_2$ , in which case it is almost GGM.*  $\square$

## 8. ORDERABILITY AND ORDER PRIMITIVITY

A partial order on a set  $S$  is said to be *trivial* if it is the equality relation on  $S$ . By a *(partially) ordered semigroup* we mean a semigroup endowed with a stable partial ordering. A *pseudovariety of ordered semigroups* is a nonempty class of finite ordered semigroups which is closed under taking images under order-preserving homomorphisms, subsemigroups with the induced ordering, and finite direct products (under the component-wise ordering). When we talk about the pseudovariety of semigroups generated by

a class  $\mathcal{C}$  of finite ordered semigroups, we mean the pseudovariety of semigroups generated by the members of  $\mathcal{C}$ , for which the order is ignored. On the other hand, every semigroup can be viewed as an ordered semigroup for the trivial ordering, reduced to equality. For a pseudovariety  $\mathbf{V}$  of semigroups, the pseudovariety of ordered semigroups  $\mathbf{V}_o$  it generates consists precisely of the members of  $\mathbf{V}$  under all possible stable partial orders. It is common practice in the literature to identify  $\mathbf{V}$  with  $\mathbf{V}_o$ .

From the point of view of the applications in computer science, the interest in pseudovarieties of ordered semigroups stems from the fact that the corresponding *positive varieties* of regular languages are defined similarly to Eilenberg's varieties of languages by dropping the requirement of closure under complementation from the definition of varieties of languages. This has prompted the investigation of many pseudovarieties of ordered semigroups and it is natural to ask what pseudovarieties of semigroups they generate. The origin of the work reported in this paper lays precisely at an attempt to answer this kind of question. Other than the application of some of the representation results from previous sections, this section contains only elementary observations. Its main purpose is to show that several familiar pseudovarieties cannot be obtained in that way.

We say that a topological semigroup is *orderable* if it admits a nontrivial closed stable partial order. The following is a simple extension to the profinite case of a well-known property of finite groups.

**Lemma 8.1.** *No profinite group is orderable.*

*Proof.* Let  $G$  be a profinite group and let  $\leq$  be a closed stable partial order on  $G$ . Let  $g \in G$  be such that  $1 \leq g$ . By stability of the partial order, the relation  $g^n \leq g^{n+1}$  holds for every positive integer  $n$ . Hence, the inequality  $g \leq g^n$  holds for every positive integer  $n$ . Considering in particular the relations  $g \leq g^{n!}$ , we deduce that, since the order is closed and  $\lim g^{n!} = 1$ , the relation  $g \leq 1$  also holds. Since  $\leq$  is assumed to be a partial order, it follows that  $g = 1$ . It follows that the relation  $\leq$  is trivial.  $\square$

In contrast with Lemma 8.1, we have the following simple observation.

**Lemma 8.2.** *Let  $\mathbf{V}$  be a pseudovariety which is not contained in  $\mathbf{G}$ . Then  $\overline{\Omega}_{\{a\}}\mathbf{V}$  is orderable.*

*Proof.* Consider the relation  $\leq$  defined by  $u \leq v$  if either  $u = v$ , or  $u = a^n$  and  $v = a^{\omega+n}$ , where  $n$  is a positive integer. One can easily check that it is a closed stable partial order on  $\overline{\Omega}_{\{a\}}\mathbf{V}$ . By hypothesis, it is nontrivial.  $\square$

For the sequel, we need the following simple auxiliary lemma.

**Lemma 8.3** ([41, Lemma 4.6.23]). *Let  $S$  be a nontrivial GGM profinite semigroup, with minimum ideal  $K$ . Then, given distinct elements  $s, t \in S$ , there exist  $x, y \in K$  such that  $xsy \neq xty$ . In particular, the maximal subgroups of  $K$  are nontrivial.*

We say that the pseudovariety  $\mathbf{V}$  is *almost unorderable* if the semigroups  $\overline{\Omega}_A\mathbf{V}$  are unorderable for finite alphabets  $A$  with  $|A|$  arbitrarily large. If  $\overline{\Omega}_A\mathbf{V}$  is unorderable for every finite set  $A$  with at least two elements, then we say that  $\mathbf{V}$  is *unorderable*. We say  $\mathbf{V}$  is *strictly orderable* if, for every



finite nonempty set  $A$ ,  $\overline{\Omega}_A \mathbf{V}$  admits a nontrivial closed stable partial order. Thus, a strictly orderable pseudovariety is not almost unorderable. We do not know if the converse is true.

The next proposition relates unorderability with the GGM property.

**Proposition 8.4.** *Let  $S$  be a nontrivial GGM profinite semigroup. Then  $S$  is unorderable.*

*Proof.* Suppose that  $\leq$  is a closed stable partial order on  $S$  for which there are elements  $s, t \in S$  such that  $s < t$ . By Lemma 8.3, the minimum ideal of  $S$  contains elements  $x$  and  $y$  such that  $xsy \neq xty$ . Since the relation  $\leq$  is stable, it follows that  $xsy < xty$ . Hence the induced order on the maximal subgroup  $H$  of  $S$  containing both  $xsy$  and  $xty$  is a nontrivial closed stable partial order on the profinite group  $H$ , which contradicts Lemma 8.1. Thus,  $S$  is unorderable.  $\square$

The following is an immediate corollary of Proposition 8.4.

**Corollary 8.5.** (a) *Every GGM pseudovariety is unorderable.*

(b) *Every almost GGM pseudovariety is almost unorderable.*  $\square$

Combining Corollary 8.5 with results from other sections, we obtain many familiar examples of unorderable pseudovarieties.

**Corollary 8.6.** *Let  $\mathbf{H}$  be a nontrivial pseudovariety of groups. Then the pseudovarieties  $\mathbf{H}$ ,  $\bar{\mathbf{H}}$ ,  $\mathbf{CS} \cap \bar{\mathbf{H}}$  are unorderable. So is  $\mathbf{CR} \cap \bar{\mathbf{H}}$ , except in the case of  $\mathbf{H} = \mathbf{Ab}_2$ , for which it is almost unorderable.*

*Proof.* In each case, it suffices to justify that the pseudovariety in question is GGM, or almost GGM in the exceptional case. For  $\mathbf{H}$ , this is obvious, but the unorderability also follows directly from Lemma 8.1. For  $\bar{\mathbf{H}}$ , the GGM property is given by Corollary 5.8. For  $\mathbf{CR} \cap \bar{\mathbf{H}}$ , it suffices to invoke Corollary 7.6. For  $\mathbf{CS} \cap \bar{\mathbf{H}}$ , the GGM property follows from the structure theorem for free profinite semigroups over this pseudovariety, which entails that they are full of torsion. The case of  $\mathbf{H} = \mathbf{G}$  has been studied in detail in [2] but the arguments and results apply equally well by replacing  $\mathbf{G}$  by a nontrivial pseudovariety of groups  $\mathbf{H}$ .  $\square$

It should be noted that there are also important pseudovarieties which are strictly orderable. Such an example is given by the pseudovariety  $\mathbf{J}$ . The following is an easy consequence of the results of [3, Section 8.2].

**Proposition 8.7.** *The pseudovariety  $\mathbf{J}$  is strictly orderable.*

*Proof.* Let  $A$  be a finite nonempty set. For  $u, v \in \overline{\Omega}_A \mathbf{J}$ , let  $u \leq v$  if every (finite) subword of  $u$  is also a subword of  $v$ . It is routine to check that  $\leq$  is a closed stable quasi-order on  $\overline{\Omega}_A \mathbf{J}$ . By [3, Theorem 8.2.8], it is a partial order.  $\square$

There is a connection between orderability and pseudovarieties of ordered semigroups that we proceed to analyze.

We say that a pseudovariety  $\mathbf{V}$  of semigroups is *order primitive* if there is no pseudovariety of ordered semigroups properly contained in  $\mathbf{V}$  that generates  $\mathbf{V}$  as a pseudovariety of semigroups.

One of the formulations of Simon's characterization of piecewise testable languages [42] is the theorem of Straubing and Thérien [45] stating that the pseudovariety of all finite  $\mathcal{J}$ -trivial monoids is generated by the pseudovariety consisting of all finite ordered monoids satisfying the inequality  $x \leq 1$ . It is easy to deduce that  $\mathbf{J}$  is generated by the pseudovariety of all finite ordered semigroups satisfying the inequalities  $xy \leq y$  and  $yx \leq y$ . Hence,  $\mathbf{J}$  is not order primitive.

**Theorem 8.8.** *Every almost unorderable pseudovariety of semigroups is order primitive.*

*Proof.* Let  $\mathbf{V}$  be an almost unorderable pseudovariety of semigroups and let  $\mathbf{U}$  be a pseudovariety of ordered semigroups properly contained in  $\mathbf{V}$ . By the version of Reiterman's Theorem for pseudovarieties of ordered semigroups [28, 33], there is a finite alphabet  $A$  and there are distinct  $u, v \in \overline{\Omega}_A \mathbf{V}$  such that  $\mathbf{U}$  satisfies the inequality  $u \leq v$ . Since  $\mathbf{V}$  is almost unorderable, we may assume that  $\overline{\Omega}_A \mathbf{V}$  is unorderable.

Consider the relation  $\preceq$  on  $\overline{\Omega}_A \mathbf{V}$  defined by  $w \preceq z$  if  $\mathbf{U}$  satisfies the inequality  $w \leq z$ . Note that it is a closed stable quasi-order on  $\overline{\Omega}_A \mathbf{V}$ . Since  $\overline{\Omega}_A \mathbf{V}$  is unorderable, it follows that  $\preceq$  fails the only missing property to be a closed stable partial order, namely anti-symmetry. Hence, there are distinct elements  $w, z \in \overline{\Omega}_A \mathbf{V}$  such that  $w \preceq z$  and  $z \preceq w$ , that is  $\mathbf{U}$  satisfies the pseudoidentity  $w = z$ , which fails in  $\mathbf{V}$ . Thus,  $\mathbf{U}$  generates a proper subpseudovariety of semigroups of  $\mathbf{V}$ .  $\square$

Note that the two-element left-zero semigroup, with the trivial order, generates  $\mathbf{LZ}$  as a pseudovariety of ordered semigroups. A pseudovariety  $\mathbf{V}$  of ordered semigroups that generates  $\mathbf{LZ}$ , as a pseudovariety of semigroups, must contain a two-element left-zero semigroup, with some stable partial order. Since the product of two copies of this semigroup contains the two element left-zero semigroup with the trivial order, we deduce that  $\mathbf{V} = \mathbf{LZ}$ , and so  $\mathbf{LZ}$  is order primitive. Note that every partial order on a left-zero semigroup is stable. Hence  $\mathbf{LZ}$  is strictly orderable, which shows that the converse of Theorem 8.8 is false.

Combining Theorem 8.8 with Corollary 8.6, we obtain the following result.

**Corollary 8.9.** *Let  $\mathbf{H}$  be a nontrivial pseudovariety of groups. Then the pseudovarieties of the form  $\mathbf{H}$ ,  $\bar{\mathbf{H}}$ ,  $\mathbf{CR} \cap \bar{\mathbf{H}}$ , and  $\mathbf{CS} \cap \bar{\mathbf{H}}$ , are order primitive.*  $\square$

A stronger result for the pseudovarieties  $\bar{\mathbf{H}}$ ,  $\mathbf{CR} \cap \bar{\mathbf{H}}$  is obtained in Section 9, which includes the pseudovariety  $\mathbf{A}$ . The structure of the lattice of varieties of ordered bands has been completely determined by Kuřil [25]. The main ingredient is to show that every variety of ordered bands that is not a variety of bands is actually a variety of ordered normal bands, which have been completely identified by Emery [19]. One may easily deduce that the pseudovariety  $\mathbf{B} = \mathbf{CR} \cap \mathbf{A}$  is order primitive. For  $\mathbf{RB} = \mathbf{CS} \cap \mathbf{A}$ , one can easily deduce that it is order primitive from the discussion above concerning  $\mathbf{LZ}$  and the dual result for  $\mathbf{RZ}$ .

## 9. JOIN IRREDUCIBILITY

Following [41, Definition 6.1.5], we say that an element  $s$  of a lattice is *strictly finite join irreducible* (*sfji*) if, whenever  $s = t \vee u$ , either  $s = t$  or  $s = u$ ; the element  $s$  is *finite join irreducible* (*fji*) if, whenever  $s \leq t \vee u$ , either  $s \leq t$  or  $s \leq u$ . Note that fji implies sfji.

The element  $s$  of a lattice is *meet-distributive* if the equality  $s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u)$  holds for all  $t$  and  $u$  in the lattice. Note that every sfji meet-distributive element of a lattice is fji.

The lattices of concern in this paper, which are both complete, are  $\mathcal{L}(\mathbf{S})$ , of all pseudovarieties of semigroups, and  $\mathcal{L}_o(\mathbf{S})$ , of all pseudovarieties of ordered semigroups, both lattices ordered by inclusion. One can easily check that  $\mathcal{L}(\mathbf{S})$  is a complete sublattice of  $\mathcal{L}_o(\mathbf{S})$ . The above lattice notions are always to be understood here with respect to one of these lattices. Of course, for an element of  $\mathcal{L}(\mathbf{S})$ , being sfji or fji with respect to  $\mathcal{L}_o(\mathbf{S})$  are stronger properties than their counterparts within the lattice  $\mathcal{L}(\mathbf{S})$ . An example of an sfji pseudovariety which is not fji in  $\mathcal{L}(\mathbf{S})$  can be found in [41, Proposition 7.3.22].

The following two theorems open a second path to applications of the representation results of the preceding sections.

**Theorem 9.1.** *Let  $\mathbf{V}$  be an almost WGGM pseudovariety of semigroups such that at least one of the following conditions holds:*

- (i)  $\mathbf{V} \bullet \mathbf{Sl} = \mathbf{V}$ ;
- (ii)  $\mathbf{V}$  contains  $\mathbf{Sl}$  and  $\mathbf{V} \bullet \mathbf{Ab}_p = \mathbf{V}$  for some prime  $p$ .

*Then  $\mathbf{V}$  is fji in the lattice  $\mathcal{L}_o(\mathbf{S})$ .*

*Proof.* Let  $\mathbf{U}$  and  $\mathbf{W}$  be pseudovarieties of ordered semigroups and suppose that  $\mathbf{V}$  is contained in  $\mathbf{U} \vee \mathbf{W}$  but in neither  $\mathbf{U}$  nor  $\mathbf{W}$ . Since pseudovarieties of ordered semigroups are defined by inequalities, there is a finite alphabet  $A$  and there are pseudowords  $u_1, u_2, w_1, w_2 \in \overline{\Omega}_A \mathbf{S}$  such that the inequality  $u_1 \leq u_2$  holds in  $\mathbf{U}$ ,  $w_1 \leq w_2$  holds in  $\mathbf{W}$ , and both inequalities (and therefore also the pseudoidentities  $u_1 = u_2$  and  $w_1 = w_2$ ) fail in  $\mathbf{V}$ .

Without loss of generality, we may assume that the sets  $c(u_1) \cup c(u_2)$  and  $c(w_1) \cup c(w_2)$  are disjoint and do not contain the letter  $z \in A$  and, furthermore, that  $\overline{\Omega}_A \mathbf{V}$  is WGGM: otherwise, we rewrite one of the pseudoidentities in a new, disjoint, alphabet, and add it to  $A$  together with enough extra letters, including  $z$ , using the hypothesis that  $\mathbf{V}$  is almost WGGM. Let  $B = A \setminus \{z\}$  and let  $\pi : \overline{\Omega}_A \mathbf{S} \rightarrow \overline{\Omega}_A \mathbf{V}$  be the natural continuous homomorphism, mapping free generators to themselves.

Since the pseudoidentity  $u_1 = u_2$  fails in  $\mathbf{V}$ , there exists a continuous homomorphism  $\varphi : \overline{\Omega}_B \mathbf{S} \rightarrow S$  into a semigroup  $S \in \mathbf{V}$  such that  $\varphi(u_1) \neq \varphi(u_2)$ . Let  $U_1$  be the two-element semilattice,  $\xi : S^1 \rightarrow U_1$  be the mapping that sends  $\varphi(u_1)$  to 0 and every other element to 1, and  $\psi$  be the extension of  $\varphi$  to a continuous homomorphism  $\overline{\Omega}_A \mathbf{S} \rightarrow M(S, U_1, \xi)$  that maps  $z$  to  $(1, 1, 1)$ . Then  $\psi(zu_1z) = (1, 0, 1) \neq (1, 1, 1) = \psi(zu_2z)$  and so the pseudoidentity  $zu_1z = zu_2z$  fails in  $\mathbf{V} \bullet \mathbf{Sl}$ . Similarly, simply replacing  $U_1$  by the additive group  $\mathbb{Z}/p\mathbb{Z}$ , the pseudoidentity  $zu_1z = zu_2z$  fails in  $\mathbf{V} \bullet \mathbf{Ab}_p$ . Thus, under the hypotheses of the theorem, the pseudoidentity  $zu_1z = zu_2z$  fails in  $\mathbf{V}$ , and the same argument and conclusion applies to the pseudoidentity  $zw_1z = zw_2z$ .

Because they all have proper content, none of the pseudowords  $\pi(zu_1z)$ ,  $\pi(zu_2z)$ ,  $\pi(zw_1z)$ ,  $\pi(zw_2z)$  belongs to the minimum ideal  $K$  of  $\overline{\Omega}_A\mathbf{V}$ . Since  $\overline{\Omega}_A\mathbf{V}$  is WGGM and  $\pi$  maps the minimum ideal  $I$  of  $\overline{\Omega}_A\mathbf{S}$  onto  $K$  [41, Lemma 4.6.10], there exist  $s, t \in I$  such that  $\pi(szu_1z) \neq \pi(szu_2z)$  and  $\pi(zw_1zt) \neq \pi(zw_2zt)$ . Let  $u'_i = szu_i z$  and  $w'_i = zw_i zt$  for  $(i = 1, 2)$ . Note that the relations  $u'_1 \mathcal{R} u'_2$  and  $w'_1 \mathcal{L} w'_2$  hold. Moreover, by Green's Lemma, the inequalities  $w'_2 u'_1 \leq w'_2 u'_2$  and  $w'_1 u'_2 \leq w'_2 u'_2$  are also nontrivial in  $\mathbf{V}$ , while they remain valid respectively in  $\mathbf{U}$  and  $\mathbf{W}$ . Let  $v = w'_2 u'_2$ . Furthermore, multiplying  $w'_2 u'_1 \leq v$  on the left by  $v^{\omega-1}$  and  $w'_1 u'_2 \leq v$  on the right by  $v^{\omega-1}$  we obtain the pseudowords

$$(12) \quad u = v^{\omega-1} w'_2 u'_1 = (zw_2 ztszu_2 z)^{\omega-1} zw_2 ztszu_1 z$$

$$(13) \quad e = v^\omega = (zw_2 ztszu_2 z)^\omega$$

$$(14) \quad w = w'_1 u'_2 v^{\omega-1} = zw_1 ztszu_2 z (zw_2 ztszu_2 z)^{\omega-1}$$

such that the following conditions hold:

$$(15) \quad u \mathcal{R} e \mathcal{L} w, \quad e^2 = e,$$

$$(16) \quad \mathbf{U} \models u \leq e, \quad \mathbf{W} \models w \leq e,$$

$$(17) \quad \pi(u) \neq \pi(e) \neq \pi(w).$$

We claim that, under the assumption that the condition (ii) holds, so does the following:

$$(18) \quad \pi(wu) \neq \pi(e).$$

To prove the claim, consider a prime  $p$  such that  $\mathbf{V} \bullet \mathbf{Ab}_p = \mathbf{V}$ . By the choice of the pseudowords  $u_1, u_2, w_1, w_2$ , there exists a continuous homomorphism  $\varphi : \overline{\Omega}_B\mathbf{S} \rightarrow S$  into a semigroup  $S \in \mathbf{V}$  such that

$$(19) \quad \varphi(w_1) \notin \{\varphi(u_1), \varphi(u_2), \varphi(w_2)\}.$$

Note that, since  $\mathbf{V}$  contains the semilattice  $U_1$ , whence the semigroup  $S \times U_1$ , we may assume that  $\varphi(w_1) \neq 1$ . Let  $\xi : S^1 \rightarrow \mathbb{Z}/p\mathbb{Z}$  map  $\varphi(w_1)$  to 1 and every other element to 0 and let  $\psi : \overline{\Omega}_A\mathbf{S} \rightarrow M(S, \mathbb{Z}/p\mathbb{Z}, \xi)$  be the extension of  $\varphi$  to a continuous homomorphism that maps  $z$  to  $(1, 0, 1)$ . Since  $\psi(z)$  is an idempotent and  $u, w, e$  start and end with  $z$ , the values of  $wu$  and  $e$  under  $\psi$  are both of the form  $(1, g, 1)$ . Since  $\xi(\varphi(w_1)) = 1$ ,  $\psi(zw_1z)$  is  $(1, 1, 1)$  while, by (19), we have  $\psi(zu_i z) = \psi(zw_2 z) = (1, 0, 1)$  ( $i = 1, 2$ ). Let  $\psi(ztsz) = (1, h, 1)$ . Then, taking into account the expressions (12)–(14) and the fact that  $\psi$  is a continuous homomorphism, we may compute

$$\psi(wu) = (1, 1, 1)(1, h, 1)^\omega = (1, 1, 1) \neq (1, 0, 1) = (1, h, 1)^\omega = \psi(e).$$

This establishes the claim since  $M(S, \mathbb{Z}/p\mathbb{Z}, \xi)$  belongs to  $\mathbf{V} \bullet \mathbf{Ab}_p$  and, therefore, to  $\mathbf{V}$ .

Consider next the following inequality, where  $y$  is a new letter:

$$(20) \quad y(uy)^{\omega-1} wuy(wy)^{\omega-1} \leq y(uy)^\omega (ey)^{\omega-1} (wy)^\omega.$$

Let  $C = A \cup \{y\}$ . We claim that (20) holds in both  $\mathbf{U}$  and  $\mathbf{W}$ . Since the arguments for the two pseudovarieties are dual, we treat only the case of  $\mathbf{U}$ .

In view of (16),  $\mathbf{U}$  satisfies the inequality  $wu \leq we$  and thus also  $wu \leq w$  because  $we = w$  by (15). Hence,  $\mathbf{U}$  satisfies the following inequalities:

$$y(uy)^{\omega-1}wuy(wy)^{\omega-1} \leq y(uy)^{\omega}(uy)^{\omega-1}wy(wy)^{\omega-1} \leq y(uy)^{\omega}(ey)^{\omega-1}(wy)^{\omega}, \quad (16)$$

which establishes that (20) holds in  $\mathbf{U}$ . We will reach a contradiction by showing that (20) does not hold in  $\mathbf{V}$ , which is contrary to the assumption that  $\mathbf{V} \subseteq \mathbf{U} \vee \mathbf{W}$ .

Suppose first that condition (18) holds, which we have not proved under the hypothesis (i). Taking also into account (17), it follows that there is a continuous homomorphism  $\varphi : \overline{\Omega}_A S \rightarrow S$  into a semigroup  $S \in \mathbf{V}$  such that  $\varphi(e) \notin \{\varphi(u), \varphi(w), \varphi(wu)\}$ . Again, we may assume that  $\varphi(e) \neq 1$ .

Let the function  $\xi : S^1 \rightarrow U_1$  map  $\varphi(e)$  to 0 and every other element to 1. Consider the extension of  $\varphi$  to a continuous homomorphism  $\psi : \overline{\Omega}_C S \rightarrow M(S, U_1, \xi)$  that sends  $y$  to  $(1, 1, 1)$ . Then we may compute

$$\begin{aligned} \psi(y(uy)^{\omega-1}wuy(wy)^{\omega-1}) &= (1, \xi(\varphi(u))\xi(\varphi(wu))\xi(\varphi(w)), 1) = (1, 1, 1), \\ \psi(y(uy)^{\omega}(ey)^{\omega-1}(wy)^{\omega}) &= (1, \xi(\varphi(u))\xi(\varphi(e))\xi(\varphi(w)), 1) = (1, 0, 1). \end{aligned}$$

Since  $M(S, U_1, \xi) \in \mathbf{V} \bullet \mathbf{Sl}$ , under the hypothesis that the condition (i) holds we deduce that the inequality (20) is not valid in  $\mathbf{V}$ .

On the other hand, if  $p$  is a prime such that  $\mathbf{V} \bullet \mathbf{Ab}_p = \mathbf{V}$ , then we consider the additive group  $\mathbb{Z}/p\mathbb{Z}$  and the mapping  $\delta : S^1 \rightarrow \mathbb{Z}/p\mathbb{Z}$  that sends  $\varphi(e)$  to 1 and every other element to 0. Now, for the extension of  $\varphi$  to a continuous homomorphism  $\chi : \overline{\Omega}_C S \rightarrow M(S, \mathbb{Z}/p\mathbb{Z}, \delta)$  that sends  $y$  to  $(1, 0, 1)$ , we may compute

$$\begin{aligned} \chi(y(uy)^{\omega-1}wuy(wy)^{\omega-1}) &= (1, -\delta(u) + \delta(wu) - \delta(w), 1) = (1, 0, 1), \\ \chi(y(uy)^{\omega}(ey)^{\omega-1}(wy)^{\omega}) &= (1, -\delta(e), 1) = (1, -1, 1). \end{aligned}$$

Since  $M(S, \mathbb{Z}/p\mathbb{Z}, \delta) \in \mathbf{V} \bullet \mathbf{Ab}_p = \mathbf{V}$ , it follows that the inequality (20) is not valid in  $\mathbf{V}$ , which again contradicts the assumption that  $\mathbf{V} \subseteq \mathbf{U} \vee \mathbf{W}$ .

It remains to treat the case where  $\pi(wu) = \pi(e)$  under the hypothesis (i). Then the set  $\pi(\{e, u, w, wu\})$  is contained in a maximal subgroup of  $K$ , which must therefore be nontrivial.<sup>2</sup> In this case, we may further replace  $u$  by  $ue$  and  $w$  by  $ew$  without affecting any of the conditions (15)–(17) and so we may assume that the pseudowords  $e, u, w$  lie in the same subgroup  $H$  of  $I$ . Since  $\pi(wu) = \pi(e)$ , then  $\pi(w) = \pi(u^{\omega-1})$  and we could replace  $u$  by  $u^{\omega-1}$  without affecting the conditions (15)–(17). Thus, we may assume that  $\pi(u) = \pi(w)$ .

For a pseudovariety  $\mathbf{X}$ , consider the relation on  $\overline{\Omega}_A S$  defined by

$$p \preceq_{\mathbf{X}} q \quad \text{if } \mathbf{X} \models p \leq q.$$

Note that  $\preceq_{\mathbf{X}}$  is a stable quasi-order. The induced relation  $\preceq$  on the profinite group  $H$  is in fact a nontrivial closed stable quasi-order. The binary relation given by  $\equiv = \preceq \cap \succeq$  is therefore a closed congruence on the profinite group  $H$  and  $\preceq$  induces a closed stable partial order on the quotient  $H/\equiv$ , which

<sup>2</sup>The reader interested only in the applications in Corollary 8.6 and its sequel may skip the remainder of the proof, since the single application for which only the hypothesis (i) can be used that we have in mind is the pseudovariety  $\mathbf{A}$ .

is a profinite group under the quotient topology. By Lemma 8.1, it follows that the induced order on the group  $H/\equiv$  is trivial, that is the relation  $\preceq$  on  $H$  is the congruence  $\equiv$ . Thus, we must have  $U \models u = e$  and  $W \models w = e$ .

From the conclusion of the preceding paragraph, we deduce that  $U \vee W$ , and therefore also  $V$ , satisfies the pseudoidentity

$$(21) \quad (uy)^\omega (wy)^\omega = (uy)^\omega (ey)^\omega (wy)^\omega.$$

However, a calculation in the semigroup  $M(S, U_1, \xi)$  considered above shows that  $\psi((uy)^\omega (vy)^\omega) = (\varphi(u), 1, 1)$ , while  $\psi((uy)^\omega (ey)^\omega (vy)^\omega) = (\varphi(u), 0, 1)$ , so that the pseudoidentity (21) fails in  $V \bullet \mathbf{Sl}$ . Hence, under the hypothesis that  $V$  satisfies condition (i), we obtain a contradiction, which completes the proof of Theorem 9.1.  $\square$

As examples of application of Theorem 9.1, taking into account the previous WGGM results (namely Corollaries 4.4, 6.3, and Theorem 7.2) and Proposition 5.2, we obtain the join irreducibility of many familiar pseudovarieties.

**Corollary 9.2.** *The pseudovarieties  $A$ ,  $\bar{H}$ ,  $DS \cap \bar{H}$ , and  $CR \cap \bar{H}$  are fji in the lattice  $\mathcal{L}_o(S)$  for every nontrivial pseudovariety of groups  $H$ .*  $\square$

That the pseudovarieties of the form  $\bar{H}$  are sfji in  $\mathcal{L}(S)$  for  $H$  an extension closed pseudovariety of groups was first proved in [26]; this is in fact deduced from the stronger property that such a pseudovariety  $\bar{H}$  cannot be expressed as a Mal'cev product of proper subpseudovarieties, which also entails the similar property for semidirect product. In [41, Corollary 7.4.23], it was proved the finite join irreducibility in  $\mathcal{L}(S)$  of the pseudovarieties of the forms  $\bar{H}$ ,  $DS \cap \bar{H}$ , and  $CR \cap \bar{H}$ , where  $H$  is a pseudovariety of groups containing some non-nilpotent group. In [8], we improved the results from [26] by showing that, for an arbitrary pseudovariety of groups  $H$ , if a pseudovariety of the form  $\bar{H}$  can be covered by a Mal'cev product of pseudovarieties then at least one of them must contain  $\bar{H}$ , which again entails the similar property for semidirect product and join.

Another sufficient condition for join irreducibility is provided by the following result.

**Theorem 9.3.** *Let  $V$  be a pseudovariety closed under concatenation that contains some nontrivial group. Then  $V$  is fji in the lattice  $\mathcal{L}_o(S)$ .*

*Proof.* We first note that  $V$  is WGGM by Corollary 4.8. The proof now follows basically the same steps as the above proof of Theorem 9.1, with appropriate modifications when the closure properties assumed in the hypothesis of Theorem 9.1 are invoked. We therefore adopt the same notation without further comment.

The first modification concerns the justification of the fact that the pseudoidentities  $zu_1z = zu_2z$  and  $zw_1z = zw_2z$  fail in  $V$ , which is immediate from the hypothesis that  $u_1 = u_2$  and  $w_1 = w_2$  fail in  $V$ , taking into account that  $V$  is equidivisible.

The second modification which is needed is to justify the inequality (18). This is done again by invoking equidivisibility of  $V$  and observing that  $wu$  admits  $zw_1z$  as a prefix, whereas  $zw_2z$  is a prefix of  $e$ , where  $z$  is a letter that does not occur in  $w_1$  and  $w_2$ .

Finally, it remains to show that the hypothesis that the inequality (20) holds in  $\mathbf{V}$  leads to a contradiction. Since  $\mathbf{V}$  is a pseudovariety of semigroups, that hypothesis means that we have the following two factorizations of the same element of  $\overline{\Omega}_C\mathbf{V}$ , where we write  $\bar{r}$  for  $\pi(r)$  with  $r \in \overline{\Omega}_A\mathbf{S}$ :

$$(22) \quad y(\bar{u}y)^{\omega-1} \cdot \bar{w}\bar{u}y(\bar{w}y)^{\omega-1} = y(\bar{u}y)^\omega \cdot (\bar{e}y)^{\omega-1}(\bar{w}y)^\omega.$$

By equidivisibility of  $\overline{\Omega}_C\mathbf{V}$ , there is some  $q \in (\overline{\Omega}_C\mathbf{V})^1$  such that one of the following conditions holds:

$$(23) \quad y(\bar{u}y)^{\omega-1}q = y(\bar{u}y)^\omega \quad \text{and} \quad \bar{w}\bar{u}y(\bar{w}y)^{\omega-1} = q(\bar{e}y)^{\omega-1}(\bar{w}y)^\omega,$$

$$(24) \quad y(\bar{u}y)^{\omega-1} = y(\bar{u}y)^\omega q \quad \text{and} \quad q\bar{w}\bar{u}y(\bar{w}y)^{\omega-1} = (\bar{e}y)^{\omega-1}(\bar{w}y)^\omega.$$

By hypothesis,  $\mathbf{V}$  contains some additive group of the form  $\mathbb{Z}/p\mathbb{Z}$ , where  $p$  is a prime. Let  $m \in \{0, 1, \dots, p-1\}$  be such that  $\varphi(q) = m$ , where  $\varphi : \overline{\Omega}_C\mathbf{V} \rightarrow \mathbb{Z}/p\mathbb{Z}$  is the unique continuous homomorphism that maps  $y$  to 1 and every other element of  $C$  to 0. From the first equalities in (23) and (24), we deduce that we must have, respectively,  $m = 1$  and  $m = p-1$ . In particular,  $y$  occurs at least once in  $q$ . Consider the unique factorization of the form  $q = q_0 y q_1$ , where  $y \notin c(q_0)$  and  $q_0, q_1 \in (\overline{\Omega}_C\mathbf{V})^1$ , where existence follows from compactness, and uniqueness from equidivisibility. From the second equalities in (23) and (24) and equidivisibility of  $\overline{\Omega}_C\mathbf{V}$ , we deduce, respectively, that  $q_0 = \bar{w}\bar{u}$  and  $q_0 = \bar{e}$ .

Suppose first that the equalities (23) hold. Consider the sequence of pseudowords  $(y(\bar{u}y)^{n!})_n$ , which converges to  $y(\bar{u}y)^\omega = y(\bar{u}y)^{\omega-1} \cdot q_0 y \cdot q_1$ . Since the multiplication in  $\overline{\Omega}_C\mathbf{V}$  is an open mapping and  $\overline{\Omega}_C\mathbf{V}$  is compact, and taking again into account that  $\overline{\Omega}_C\mathbf{V}$  is equidivisible, there are sequences of positive integers  $(j_i)_i$ ,  $(k_i)_i$ , and  $(\ell_i)_i$  such that  $\lim y(\bar{u}y)^{j_i} = y(\bar{u}y)^{\omega-1}$ ,  $\lim (\bar{u}y)^{k_i} = q_0 y$ ,  $\lim (\bar{u}y)^{\ell_i} = q_1$ , and  $(j_i + k_i + \ell_i)_i$  is a strictly increasing sequence of factorials. Since  $y$  does not occur in  $q_0$ , it follows that  $q_0 = \bar{u}$ . Since  $\bar{w}\bar{u} = q_0$ , by the preceding paragraph, we obtain  $\bar{w}\bar{u} = \bar{u}$ , which contradicts (17) by Green's Lemma. The case where the equalities (24) hold is handled similarly.  $\square$

Theorem 9.3 applies in particular to pseudovarieties of the form  $\bar{\mathbf{H}}$ , with  $\mathbf{H}$  a nontrivial pseudovariety of groups, but the conclusion is already part of Corollary 9.2. A new result is obtained by combining Theorem 9.3 with Proposition 2.4 and Corollary 4.6, the case  $n = 0$  being given by Corollary 9.2.

**Corollary 9.4.** *For every pseudovariety of groups  $\mathbf{H}$ , the pseudovarieties  $\mathbf{C}_n \cap \bar{\mathbf{H}}$  are fji in the lattice  $\mathcal{L}_0(\mathbf{S})$ .*  $\square$

Corollary 9.4 implies, in particular, that the complexity pseudovarieties  $\mathbf{C}_n$  are fji in the lattice  $\mathcal{L}(\mathbf{S})$ , which solves the first part of [41, Problem 43].

We conclude with a connection between sfji and order primitivity. For a pseudovariety  $\mathbf{U}$  of order semigroups, its *order dual* is the pseudovariety  $\mathbf{U}^d$  consisting of the ordered semigroups  $(S, \leq)$  such that  $(S, \geq)$  belongs to  $\mathbf{U}$ .

**Lemma 9.5** ([34]). *Let  $\mathbf{V}$  be a pseudovariety of semigroups and let  $\mathbf{U}$  be a pseudovariety of ordered semigroups contained in  $\mathbf{V}$ . Then  $\mathbf{U}$  generates  $\mathbf{V}$  as a pseudovariety of semigroups if and only if  $\mathbf{V} = \mathbf{U} \vee \mathbf{U}^d$ .*

pseudovariety	WGGM	GGM	in $\mathcal{L}(S)$		order- primitive	in $\mathcal{L}_0(S)$	
			sfji	fji		sfji	fji
$A$	Y	N	Y	Y	Y	Y	Y
$\bar{H} (I \neq H \subseteq G)$	Y	Y	Y	Y	Y	Y	Y
$C_n \cap \bar{H} (I \neq H \subseteq G)$	Y	?	Y	Y	Y	Y	Y
$DS \cap \bar{H} (I \neq H \subseteq G)$	Y	?	Y	Y	Y	Y	Y
$DO \cap \bar{H} (H \subseteq G)$	Y	N	?	?	?	?	?
$DS \cap C_n (n \geq 1)$	Y	?	?	?	?	?	?
$B$	almost	N	Y	Y	Y	Y	?
$CR \cap \bar{H} (I \neq H \subseteq G)$	Y	almost	Y	Y	Y	Y	Y
$CR \cap C_n (n \geq 1)$	Y	?	?	?	?	?	?

TABLE 3. Summary of results and open problems

The following is an immediate application of Lemma 9.5.

**Proposition 9.6.** *Every pseudovariety of semigroups which is sfji in the lattice  $\mathcal{L}_o(S)$  is order primitive.*  $\square$

Combining Proposition 9.6 with Corollaries 9.2 and 9.4, we obtain the following result.

**Corollary 9.7.** *Let  $H$  be a nontrivial pseudovariety of groups. Then the pseudovarieties  $A$ ,  $C_n \cap \bar{H}$ ,  $\bar{H}$ ,  $DS \cap \bar{H}$ , and  $CR \cap \bar{H}$  are order primitive.*  $\square$

Interesting related pseudovarieties which do not fall in the realm of application of Theorem 9.1 are those of the form  $DO \cap \bar{H}$ , in particular  $DA = DO \cap A = DS \cap A$ . We do not know whether  $DA$  is fji or at least sfji, within  $\mathcal{L}_o(S)$  or even within  $\mathcal{L}(S)$ , or order primitive. On the other hand, the pseudovariety  $B = CR \cap \bar{H}$  is well known to be sfji in  $\mathcal{L}(S)$ , a result that is an immediate consequence of the structure of the lattice of varieties of bands (see for instance [3, Section 5.5] for a diagram of the lattice and bibliographic references). Taking into account that  $B$  is meet distributive in  $\mathcal{L}(S)$  [36], it follows that  $B$  is fji. By the discussion at the end of Section 8, it follows that  $B$  is also sfji in  $\mathcal{L}_o(S)$ .

Table 3 summarizes the results and questions about the various pseudovarieties of concern in this paper. For each pair pseudovariety, property, Y/N indicates whether or not the pseudovariety enjoys the property, a question mark indicates that the answer is presently unknown to the authors, and the word *almost* has the technical meaning introduced in Section 3.2.

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## REFERENCES

- [1] D. Albert, R. Baldinger, and J. Rhodes, *The identity problem for finite semigroups (the undecidability of)*, J. Symbolic Logic **57** (1992), 179–192.
- [2] J. Almeida, *On finite simple semigroups*, Proc. Edinburgh Math. Soc. **34** (1991), 205–215.
- [3] ———, *Finite semigroups and universal algebra*, World Scientific, Singapore, 1995, English translation.
- [4] ———, *A syntactical proof of locality of DA*, Int. J. Algebra Comput. **6** (1996), 165–177.
- [5] J. Almeida and A. Costa, *Infinite-vertex free profinite semigroupoids and symbolic dynamics*, J. Pure Appl. Algebra **213** (2009), 605–631.
- [6] ———, *Handbook of AutoMathA*, ch. Profinite topologies, European Math. Soc. Publ. House, 2015, To appear.
- [7] ———, *A note on pseudovarieties of completely regular semigroups*, Bull. Austral. Math. Soc. (2015), Published online, doi:10.1017/S0004972715000532.
- [8] J. Almeida and O. Klíma, *On the irreducibility of pseudovarieties of semigroups*, Tech. Report CMUP 2011-33, Masaryk Univ. and Univ. Porto, 2011.
- [9] J. Almeida and P. G. Trotter, *Hyperdecidability of pseudovarieties of orthogroups*, Glasgow Math. J. **43** (2001), 67–83.
- [10] ———, *The pseudoidentity problem and reducibility for completely regular semigroups*, Bull. Austral. Math. Soc. **63** (2001), 407–433.
- [11] J. Almeida and M. V. Volkov, *Subword complexity of profinite words and subgroups of free profinite semigroups*, Int. J. Algebra Comput. **16** (2006), 221–258.
- [12] K. Auinger and B. Steinberg, *On the extension problem for partial permutations*, Proc. Amer. Math. Soc. **131** (2003), 2693–2703.
- [13] J.-C. Birget, *Iteration of expansions—unambiguous semigroups*, J. Pure Appl. Algebra **34** (1984), 1–55.
- [14] J.-C. Birget and J. Rhodes, *Almost finite expansions of arbitrary semigroups*, J. Pure Appl. Algebra **32** (1984), 239–287.
- [15] J. H. Carruth, J. A. Hildebrandt, and R. J. Koch, *The theory of topological semigroups. Vol. 2*, Pure and Applied Mathematics, no. 100, Marcel Dekker, New York, 1986.
- [16] L. Chaubard, J.-E. Pin, and H. Straubing, *Actions, wreath products of C-varieties and concatenation product*, Theor. Comp. Sci. **356** (2006), 73–89.
- [17] V. Diekert, M. Kufleitner, and P. Weil, *Star-free languages are Church-Rosser congruential*, Theor. Comp. Sci. (2012), 129–135.
- [18] S. Eilenberg, *Automata, languages and machines*, vol. B, Academic Press, New York, 1976.
- [19] S. J. Emery, *Varieties and pseudovarieties of ordered normal bands*, Semigroup Forum **58** (1999), 348–366.
- [20] R. Engelking, *General topology*, Sigma Series in Pure Mathematics, no. 6, Heldermann Verlag Berlin, 1989, Revised and completed edition.
- [21] K. Henckell, S. Margolis, J.-E. Pin, and J. Rhodes, *Ash’s type II theorem, profinite topology and Malcev products. Part I*, Int. J. Algebra Comput. **1** (1991), 411–436.
- [22] J. A. Hildebrandt, J. D. Lawson, and D. P. Yeager, *The translational hull of a topological semigroup*, Trans. Amer. Math. Soc. **221** (1976), 251–280.
- [23] J. Kaĉourek and L. Polák, *On the word problem for free completely regular semigroups*, Semigroup Forum **34** (1986), 127–138.
- [24] K. Krohn, J. Rhodes, and B. Tilson, *Lectures on the algebraic theory of finite semigroups and finite-state machines*, Algebraic Theory of Machines, Languages, and Semigroups (New York) (M. A. Arbib, ed.), Academic Press, 1968, Chapter 6 with M. A. Arbib.
- [25] M. Kuřil, *On varieties of ordered semigroups*, Semigroup Forum **90** (2015), 475–490.
- [26] S. Margolis, M. Sapir, and P. Weil, *Irreducibility of certain pseudovarieties*, Comm. Algebra **26** (1998), 779–792.
- [27] J. D. McKnight, Jr. and A. J. Storey, *Equidivisible semigroups*, J. Algebra **12** (1969), 24–48.

- [28] V. A. Molchanov, *Nonstandard characterization of pseudovarieties*, Algebra Universalis **33** (1995), 533–547.
- [29] K. Numakura, *Theorems on compact totally disconnected semigroups and lattices*, Proc. Amer. Math. Soc. **8** (1957), 623–626.
- [30] J.-E. Pin, *Propriétés syntactiques du produit non ambigu*, Automata, languages and programming (Proc. Seventh Internat. Colloq., Noordwijkerhout, 1980) (Berlin), Lect. Notes in Comput. Sci., vol. 85, Springer, 1980, pp. 483–499.
- [31] ———, *Varieties of formal languages*, Plenum, London, 1986, English translation.
- [32] ———, *A variety theorem without complementation*, Russian Math. (Iz. VUZ) **39** (1995), 80–90.
- [33] J.-E. Pin and P. Weil, *A Reiterman theorem for pseudovarieties of finite first-order structures*, Algebra Universalis **35** (1996), 577–595.
- [34] ———, *Polynomial closure and unambiguous product*, Theory Comput. Syst. **30** (1997), 383–422.
- [35] N. R. Reilly, *The Rhodes expansion and free objects in varieties of completely regular semigroups*, J. Pure Appl. Algebra **69** (1990), 89–109.
- [36] N. R. Reilly and S. Zhang, *Complete endomorphisms of the lattice of pseudovarieties of finite semigroups*, Bull. Amer. Math. Soc. **55** (1997), 207–218.
- [37] J. Reiterman, *The Birkhoff theorem for finite algebras*, Algebra Universalis **14** (1982), 1–10.
- [38] J. Rhodes, *Infinite iteration of matrix semigroups, II. Structure theorem for arbitrary semigroups up to aperiodic morphism*, J. Algebra **100** (1986), 25–137.
- [39] J. Rhodes and D. Allen Jr., *Synthesis of the classical and modern theory of finite semigroups*, Adv. in Math. **11** (1973), 238–266.
- [40] J. Rhodes and B. Steinberg, *Join irreducible pseudovarieties, group mapping, and Kovács-Newman semigroups*, LATIN 2004: Theoretical informatics, Lecture Notes in Comput. Sci., vol. 2976, Springer, Berlin, 2004, pp. 279–291.
- [41] ———, *The  $q$ -theory of finite semigroups*, Springer Monographs in Mathematics, Springer, 2009.
- [42] I. Simon, *Piecewise testable events*, Proc. 2nd GI Conf. (Berlin), Lect. Notes in Comput. Sci., vol. 33, Springer, 1975, pp. 214–222.
- [43] H. Straubing, *Aperiodic homomorphisms and the concatenation product of recognizable sets*, J. Pure Appl. Algebra **15** (1979), 319–327.
- [44] ———, *Finite semigroup varieties of the form  $V * D$* , J. Pure Appl. Algebra **36** (1985), 53–94.
- [45] H. Straubing and D. Thérien, *Partially ordered monoids and a theorem of I. Simon*, J. Algebra **119** (1988), 393–399.

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